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# ASYMPTOTIC PROPERTIES OF THE ESTIMATOR OF THE CONDITIONAL DISTRIBUTION FOR ASSOCIATED FUNCTIONAL DATA

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**Abstract:** The purpose of the paper was to investigate by the kernel method a nonparametric estimate of the conditional density function of a scalar response variable given a random variable taking values in a separable real Hilbert space when the observations are quasi-associated dependent. Under some general conditions, the authors established the pointwise almost complete consistencies with rates of this estimator. The principal aim is the investigate the convergence rate of the proposed estimator.

**Keywords:** nonparametric estimation, small ball probability, quasi-associated data.

# 1. Introduction

In statistics, (FDA) has received much attention in the field of applied mathematics. This type of data is collected from numerous fields, such as econometrics study, epidemiology control, environmental and ecological sciences, and many other sections. Functional Statistics was published by Ferraty, Laksaci, and Vieu (2006), who obtained some properties in the case i.i.d. The study of statistical models for infinite-dimensional (functional) data has been the subject of several works in recent statistical literature; the reader can consult the studies by Akkal, Kadiri, and Rabhi (2021), Kadiri, Rabhi, Khardani, and Akkal (2021), and references therein. The recent result on nonparametric estimation was obtained by Hamri, Mekki, Rabhi, and Kadiri (2022), who introduced a kernel estimator of the conditional distribution function and proved some asymptotic properties (with rate) in various situations including censored and/or independent variables. Since then, abundant literature has appeared on the estimation of the conditional density and its derivatives, in particular to utilise it to estimate the conditional mode. Considering mixed observtions, Ferraty, Laksaci, and Vieu (2005) established the convergence (a.co) of the kernel estimator of the conditional mode defined by the random variable maximizing the conditional density. Alternatively, Ezzahrioui and Ould-Saïd (2008, 2010) estimated the conditional mode by the point that cancels the derivative of the kernel density estimator. In related disciplines, including analyses of reliability, theoretical physics, MVA, and biological sciences the associated random variables are crucial. Many studies used positive and negative dependent random variables. The association case is a type of weak dependence introduced by Bulinski (Bulinski and Suquet, 2001) for stochastic processes in R. It was generalized by Ezzahrioui and Ould-Saïd (2008) to real random fields, and it provides a unified approach to studying families of both positive dependence and negative dependence random variables. There are few of articles dealing with the nonparametric estimation of quasi-associated data. One can cite, for quasi-associated Hilbertian random variables, Douge (2010) studied its limit theorem, Attaoui et al. (2015) examined asymptotic properties for Regression M-estimator, and for a functional single index structure, while Tabti and Aït Saidi (2018) discussed the simulation and estimation part of the conditional risk function in the quasi-associated data case. The same model, the asymptotic normality, was studied by Daoudi, Mechab and Chikr Elmezouar (2020). In the case of relative regression, Mechab and Laksaci (2016) addressed the nonparametric estimation for associated r.v. For the asymptotic normality of the nonparametric conditional cumulative function estimate studied by Daoudi and Mechab (2019). The intention of this work is to check the estimator properties proposed by Ferraty et al. (2006); in cases of associated data the a.co convergence is established (rate) of a kernel estimate for the hazard function when the variable is real random conditioned by a functional explanatory variable.

## 2. The model

Consider $Z_i = (X_i, Y_i)_{1 \le i \le n}$ be a $n$ quasi-associated (Q.A) random processes, identically distributed as the random pair $Z = (X, Y)$, with values in $\mathcal{H} \times \mathbb{R}$, where $\mathcal{H}$ is a separable real Hilbert space, with the norm $\|.\|$ generated by an inner product $\langle .,. \rangle$.

The authors considered the semi-metric $d$ defined by $\forall (x, x') \in \mathcal{H} / d(x, x') = \|x - x'\|$. In the rest of this paper, it was considered that$x \in \mathcal{H}$ (fixed) and $\mathcal{N}_x$ mention a fixed neighbourhood of $x$ and $\mathcal{S} \subset \mathbb{R}$.

To estimate the conditional distribution function, let us consider the following functional kernel estimators

$$\hat{F}^x(y) = \frac{\sum_{i=1}^n K(h_K^{-1} d(x, X_i)) H(h_H^{-1}(y - Y_i))}{\sum_{i=1}^n K(h_K^{-1} d(x, X_i))}, \forall\, y \in \mathbb{R},$$

where: $K$ is a probability density function (the so-called kernel function), $H$ is a cumulative distribution function, $h_K = h_{K,n}$ (resp.$h_H = h_{H,n}$) is a sequence of positive real which converges to 0 when $n \to \infty$, with:

$$K_i(x) = K(h_K^{-1} d(x, X_i)) \text{ and } H_i(y) = H(h_H^{-1}(y - Y_i)),$$

one can write

$$\hat{F}^x(y) = \frac{\hat{F}_N(y, x)}{\hat{F}_D(x)},$$

$$\hat{F}_N(y, x) = \frac{1}{n \mathbb{E}[K_1(x)]} \sum_{i=1}^n K_i(x) H_i(y)$$

and for

$$\hat{F}_D(x) = \frac{1}{n \mathbb{E}[K_1(x)]} \sum_{i=1}^n K_i(x),$$

and $\hat{f}^x(y)$ is the conditional density estimator for $f^x(y)$ given by

$$\hat{f}^x(y) = \frac{h_K^{-1} \sum_{i=1}^n K(h_K^{-1} d(x, X_i)) H'(h_H^{-1}(y - Y_i))}{\sum_{i=1}^n K(h_K^{-1} d(x, X_i))}, \forall\, y \in \mathbb{R},$$

where $H'$ is kernel function (the first derivative of a given distribution function $H$), one can write

$$\hat{f}^x(y) = \frac{\hat{f}_N(y,x)}{\hat{F}_D(x)},$$

where

$$\hat{f}_N(y,x) = \frac{1}{nh_H \mathbb{E}[K_1(x)]} \sum_{i=1}^{n} K_i(x) H_i'(y),$$

with

$$K_i(x) = K(h_K^{-1} d(x, X_i)) \text{ and } H_i'(y) = H'(h_H^{-1}(y - Y_i)).$$

## 3. Assumptions

**Definition 3.1.** A sequence $(X_n)_{n\epsilon\mathbb{N}}$ of real random vectors variables be Quasi--Association (QA), if for any disjoint subsets $I$ and $J$ of $\mathbb{N}$ and all bounded Lipschitz functions $f\colon \mathbb{R}^{|I|d} \to \mathbb{R}$ and $g\colon \mathbb{R}^{|J|d} \to \mathbb{R}$ satisfying

$$Cov\left(f\ (X_i, i \in I), g(X_j, j \in J)\right) \leq Lip(f)Lip(g) \sum_{i\in I} \sum_{j\in J} \sum_{k=1}^{d} \sum_{l=1}^{d} \left|Cov(X_i^k, X_j^l)\right|,$$

where $X_i^k$ imply the $k^{th}$composent of $X_i$.

$$Lip(f) = sup_{x\neq y} \frac{|f(x) - f(y)|}{\|x-y\|_1} \ with \ \|(x_1, \ldots, x_k)\|_1 = |x_1| + \cdots + |x_k|.$$

Later in the paper, when there is no possible confusion, the authors refer by $\mathcal{C}$ or/and $\mathcal{C}'$ to some completely positive global constants whose values are allowed to be changed. Suppose the coefficient of covariance is defined as:

$$\lambda_k = sup_{s\geq k} \sum_{|i-j|\geq s} \lambda_{ij},$$

where

$$\lambda_{ij} = \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \left|cov(X_i^k, X_j^l)\right| + \sum_{k=1}^{\infty} \left|cov(X_i^k, Y_j)\right| + \sum_{l=1}^{\infty} \left|cov(Y_i, X_j^l)\right| + \left|cov(Y_i, Y_j)\right|,$$

$X_i^k$ imply the $k^{th}$ composent of $X_i$ specified as $X_i^k := \langle X_i, e^k \rangle$, with $h_K > 0$, let $B_\theta(x, h_K) := \{x' \in \mathcal{H} / d(x', x) < h_K\}$ a ball with its centre $x$ and radius $h_K$. Let us mention the assumptions that help us reach the desired results:

**(H1)** $\mathbb{P}\big(X \in B(x, h_K)\big) = \phi_x(h_K) > 0$ and $\beta(x,.)$ such that:

$$\forall\, s \in [0,1], \lim_{h_K \to 0} \frac{\phi_x(s h_K)}{\phi_x(h_K)} = \beta(x,\cdot).$$

**(H2)** Conditional distribution $F^x(y)$, satisfies the Hölder condition, i.e.

$$\forall (x_1, x_2) \in \mathcal{N}_x^2, \forall (y_1, y_2) \in \mathcal{S}^2,$$
$$|F^{x_1}(y_1) - F^{x_2}(y_2)| \leq C\big(d^{b_1}(x_1, x_2) + |y_1 - y_2|^{b_2}\big),\ b1 > 0,\ b2 > 0.$$

**(H3)** Kernel $H$ is a positive bounded function such that $\forall (t_1, t_2) \in \mathbb{R}^2$, $|H(t_1) - H(t_2)| \leq C|t_1 - t_2|$, $\int H'(t)dt = 1$, $\int H'^2(t)dt < \infty$ and $\int |t|^{b_2} H'(t)dt < \infty$.

**(H4)** Kernel $K$ is a positive bounded continuous Lipschitz function on [0,1] such that: $\mathcal{C}\mathbb{1}_{[0,1]}(\cdot) < K(\cdot) < \mathcal{C}'\mathbb{1}_{[0,1]}(\cdot)$, where $\mathbb{1}_{[0,1]}$ is an indicator function.

**(H5)** The sequence of random pairs $(X_i, Y_i)_{i \in \mathbb{N}}$ is quasi-associated with covariance coefficient $\lambda_k$, $k \in \mathbb{N}$ satisfying

$$\exists \alpha > 0, \exists\, \mathcal{C} > 0, \text{ such that } \lambda_k \leq \mathcal{C} e^{-\alpha k}.$$

**(H6)** $\psi_{i,j}(h) = \mathbb{P}\big[(X_i, X_j) \in \beta(x, h_K) \times \beta(x, h_K)\big] = O\big(\phi_x^2(h_K)\big)$, satisfy

$$\sup_{i \neq j} \psi_{i,j}(h) = O\big(\phi_x^2(h_K)\big) > 0.$$

**(H7)** The bandwidth $h_K$ and $h_H$, are sequences of positive numbers satisfying for $j = 0,1$:

- $\lim_{n\to\infty} h_K = 0$ *and* $\lim_{n\to\infty} h_H = 0,$
- $\lim_{n\to\infty} \big(h_H^{b_2} + h_K^{b_1}\big)\sqrt{n\phi_x(h_K)} = 0,$
- $\lim_{n\to\infty} \frac{log(n)}{n h_H^j \phi_x(h_K)} = 0$ *and* $\lim_{n\to\infty} \frac{log^5(n)}{n h_H^j \phi_x(h_K)} = 0,$

where $X_i^k$ imply the $k^{th}$ composent of $X_i$.

## 4. Main results

**Theorem 4.1.** Under assumptions **(H1)-(H7)**, we have, for any $x \in \mathcal{H}$:

$$\widehat{F}^x(y) - F^x(y) = O\big(h_K^{b_1} + h_H^{b_2}\big) + O_{a.co}\left(\sqrt{\frac{\log(n)}{n\phi_x(h_K)}}\right).$$

**Proof of Theorem 4.1:** This based on the following decomposition:

$$\hat{F}^x(y) - F^x(y) = \frac{\hat{F}_N^x(y) - F^x(y)\hat{F}_D(x)}{\hat{F}_D(x)}$$
$$= \frac{1}{\hat{F}_D^x(y)}\left(\hat{F}_N^x(y) - \mathbb{E}\left[\hat{F}_N^x(y)\right]\right)$$
$$-\frac{1}{\hat{F}_D^x(y)}\left(F^x(y) - \mathbb{E}\left[\hat{F}_N^x(y)\right]\right)$$
$$+\frac{F^x(y)}{\hat{F}_D(x)}\left\{\mathbb{E}\left[\hat{F}_D(x)\right] - \hat{F}_D(x)\right\}.$$

Finally, the proof of this theorem is a direct consequence of the following intermediate results.

**Lemma 4.1.** Under assumptions **(H1)-(H4)-(H6)**, we obtain:

$$\hat{F}_N^x(y) - \mathbb{E}\left[\hat{F}_N^x(y)\right] = O_{a.co}\left(\sqrt{\frac{\log(n)}{n\phi_x(h_K)}}\right).$$

**Corollary 4.1.** Under assumptions **(H1)-(H4)-(H6)**, we obtain:

$$\sum_{i=1}^{\infty} \mathbb{P}\left(\left|\hat{F}_D^x\right| < \frac{1}{2}\right) < \infty.$$

**Lemma 4.2.** Under assumptions **(H1)-(H3)-(H6)**, we obtain:

$$\hat{F}_D^x(y) - \mathbb{E}\left[\hat{F}_D^x(y)\right] = O_{a.co}\left(\sqrt{\frac{\log(n)}{nh_H\phi_x(h_K)}}\right).$$

**Lemma 4.3.** Under hypotheses **(H1)-(H6)**, we obtain:

$$F^x(y) - \mathbb{E}\left[\hat{F}_N^x(y)\right] = O\left(h_K^{b_1} + h_H^{b_2}\right).$$

## 5. Proof of technical lemmas

First of all, let us state the following lemmas.

**Lemma 5.1** (Douge, 2010). Let $(X_n)_{n\epsilon\mathbb{N}}$ be a quasi-associated sequence of random variables with values in $\mathcal{H}$. Let $f \in BL\left(\mathcal{H}^{|I|}\right)\cap\mathbb{L}^\infty$ and $g \in BL\left(\mathcal{H}^{|J|}\right)\cap\mathbb{L}^\infty$ for some finite disjoint subsets $I, J \in \mathbb{N}$, then

$$Cov\ \left(f\ (X_i, i \in I), g\left(X_j, j \in J\right)\right) \leq Lip(f)Lip(g)\sum_{i\in I}\sum_{j\in J}\sum_{k=1}^{d}\sum_{l=1}^{d}\left|Cov\left(X_i^k, X_j^l\right)\right|,$$

where $BL(\mathcal{H}^u, u > 0)$ is the set bounded Lipschitz functions $f: \mathcal{H}^u \longrightarrow \mathbb{R}$ and $\mathbb{L}^\infty$ is the set of bounded functions.

**Lemma 5.2** (Kallabis and Newmann, 2006). Let $X_1, \dots, X_n$be the real random such that $\mathbb{E}(X_j) = 0$ and $\mathbb{P}(|X_j| \leq M) = 1$ for all $j = 1, \dots, n$ and some $M < \infty$, let $\sigma_n^2 = Var(\sum_{i=1}^n \delta_i)$.

Assume furthermore, that there exist $K < \infty$ and $\beta > \infty$ such that, for all $u$-uplets $(s_1 \dots s_u) \in \mathbb{N}^u$, $(t_1 \dots t_v) \in \mathbb{N}^v$ with $1 \leq s_1 \leq \dots \leq s_u \leq t_1 \leq \dots \leq t_v \leq n$, the following inequality is fulfilled

$$\left|Cov\left(X_{s_1} \dots X_{s_u}, X_{t_1} \dots X_{t_v}\right)\right| \leq K^2 M^{u+v-2} v e^{-\beta(t_1 - s_u)}.$$

Then

$$\mathbb{P}\left(\left|\sum_{j=1}^n X_j\right| > t\right) \leq \exp\left\{-\frac{t^2/2}{A_n + B_n^{1/3} t^{5/2}}\right\},$$

for some

$$A_n \leq \sigma_n^2$$

and

$$B_n = \left(\frac{16nK^2}{9A_n(1 - e^{-\beta})} \vee 1\right) \frac{2(K \vee M)}{1 - e^{-\beta}}.$$

**Proof of Lemma 5.1:** Let us put

$$\delta_i = \frac{1}{n\mathbb{E}[K_1(x)]} \chi(X_i, Y_i), 1 \leq i \leq n,$$

where $X_i \in \mathcal{H}$, $Y_i \in \mathbb{R}$

$$\chi(X_i, Y_i) = K(h_K^{-1} d(x, X_i))\, H(y - Y_i) - \mathbb{E}[K_1 H_1]. \qquad (1)$$

Clearly, we have $\mathbb{E}[\delta_i] = 0$ and

$$\left|\hat{F}_N^x(y) - \mathbb{E}\left[\hat{F}_N^x(y)\right]\right| = \sum_{i=1}^n \delta_i,$$

we can also write

$$||\chi||_\infty^2 \leq 2\, \mathcal{C}\, ||K||_\infty ||H||_\infty,$$

and

$$\mathrm{Lip}(\chi) \leq \mathcal{C}(\, h_K^{-1} ||H||_\infty \mathrm{Lip}(K) + h_H^{-1} ||K||_\infty \mathrm{Lip}(H).$$

Now, to apply Lemma 5.2, evaluate the variance $Var(\sum_{i=1}^{n}\delta_i)$ and the covariance term $Cov\left(\prod_{i=1}^{u}\delta_{s_i},\prod_{j=1}^{v}\delta_{t_j}\right) = Cov\left(\delta_{s_1}\dots\delta_{s_u},\delta_{t_1}\dots\delta_{t_v}\right)$, for all $(s_1\dots s_u)\in\mathbb{N}^u$, $(t_1\dots t_v)\in\mathbb{N}^v$ with $1\le s_1\le\cdots\le s_u\le t_1\le\cdots\le t_v\le n$.

Firstly, for the covariance term, consider the following cases: If: $t_1 = s_u$. By using the fact that $\mathbb{E}[|K_1H_1|]=O(\phi_x(h_K))$ and $\mathbb{E}[|K_1|]=O(\phi_x(h_K))$ we obtain

$$\left|Cov\left(\prod_{i=1}^{u}\delta_{s_i},\prod_{j=1}^{v}\delta_{t_j}\right)\right| \le \left(\frac{\mathcal{C}}{n\mathbb{E}[K_1(x)]}\right)^{u+v} \mathbb{E}_\chi|\mathrm{X}_1,\mathrm{Y}_1|^{u+v}$$

$$\le \left(\frac{\mathcal{C}||\mathrm{K}||_\infty||\mathrm{H}||_\infty}{n\mathbb{E}[K_1(x)]}\right)^{u+v} \mathbb{E}[K_1H_1]$$

$$\le \phi_x(h_K)\left(\frac{\mathcal{C}}{n\phi_x(h_K)}\right)^{u+v}.$$

If $t_1 > s_u$, use the quasi-association, by **(H5)**, to obtain

$$\left|Cov\left(\prod_{i=1}^{u}\delta_{s_i},\prod_{j=1}^{v}\delta_{t_j}\right)\right| \le \left(\frac{h_K^{-1}\,\mathrm{Lip(K)} + h_H^{-1}\,\mathrm{Lip(H)}}{n\mathbb{E}[K_1(x)]}\right)^2$$

$$\left(\frac{\mathcal{C}}{n\mathbb{E}[K_1(x)]}\right)^{u+v}\sum_{i=1}^{u}\sum_{j=1}^{v}\lambda_{s_it_j} \tag{2}$$

$$\le (h_K^{-1}\,\mathrm{Lip(K)} + h_H^{-1}\,\mathrm{Lip(H)})^2\left(\frac{\mathcal{C}}{n\mathbb{E}[K_1(x)]}\right)^{u+v} v\lambda_{t_1-s_u}$$

$$\le \left(h_K^{-1}\,\mathrm{Lip(K)} + h_H^{-1}\,\mathrm{Lip(H)}\right)^2\left(\frac{\mathcal{C}}{\phi_x(h_K)}\right)^{u+v} ve^{-\alpha(t_1-s_u)}.$$

On the other hand, by **(H6)** we obtain

$$\left|Cov\left(\prod_{i=1}^{u}\delta_{s_i},\prod_{j=1}^{v}\delta_{t_j}\right)\right| \le \left(\frac{\mathcal{C}||\mathrm{K}||_\infty||\mathrm{H}||_\infty}{n\mathbb{E}[K_1(x)]}\right)^{u+v-2} \left(\mathbb{E}\left|\Delta_{s_u},\Delta_{t_1}\right| + \mathbb{E}\left|\Delta_{s_u}\right|\mathbb{E}\left|\Delta_{t_1}\right|\right)$$

$$\le \left(\frac{\mathcal{C}\,||\mathrm{K}||_\infty||\mathrm{H}||_\infty}{n\mathbb{E}[K_1(x)]}\right)^{u+v-2}\left(\frac{\mathcal{C}}{n\mathbb{E}[K_1(x)]}\right)^2\times$$

$$\left(\sup_{\mathrm{i}\neq\mathrm{j}}\mathbb{P}\left[(X_i,X_j)\in\ \beta(x,h_K)\times\beta(x,h_K)\right] + (\mathbb{P}[X_1\in\beta(x,h_K)])^2\right) \tag{3}$$

$$\le \left(\frac{\mathcal{C}}{h_H\phi_x(h_K)}\right)^{u+v}\left(\phi_x(h_K)\right)^2.$$

Furthermore, taking a $\gamma -$ power of (2), $(1-\gamma)-$ power of (3), with $0 < \gamma < 1$, we obtain an upper-bound of the tree terms as follows, for: $1 \leq s_1 \leq \cdots \leq s_u \leq t_1 \leq \cdots \leq t_v \leq n$,

$$\left| Cov\left( \prod_{i=1}^{u} \delta_{s_i}, \prod_{j=1}^{v} \delta_{t_j} \right) \right| \leq \phi_x(h_K) \left( \frac{C}{n\phi_x(h_K)} \right)^{u+v}.$$

Secondly, for the variance term $Var(\sum_{i=1}^{n} \delta_i)$, we put for all $1 \leq i \leq n$:

$$\begin{aligned} \left| Var\left( \sum_{i=1}^{n} \delta_i \right) \right| &= \left( \frac{1}{n\mathbb{E}[K_1(x)]} \right)^2 \sum_{i=1}^{u} \sum_{j=1}^{v} Cov\big(K_i H_i, K_j H_j\big) \\ &= \underbrace{\left( \frac{1}{n\mathbb{E}[K_1(x)]} \right)^2 Var(K_1 H_1)}_{T_1} \\ &\quad + \underbrace{\left( \frac{1}{n\mathbb{E}[K_1(x)]} \right)^2 \sum_{i=1}^{n} \sum_{j=1, i\neq j}^{n} Cov\big(K_i H_i, K_j H_j\big)}_{T_2}. \end{aligned} \tag{4}$$

For the first term **T1**, we have

$$Var(K_1 H_1) = \mathbb{E}[K_1^2 H_1^2] - (\mathbb{E}[K_1 H_1])^2,$$

then,

$$\mathbb{E}[K_1^2 H_1^2] = \mathbb{E}\left[ K_1^2 \mathbb{E}[H_1^2 | X_1] \right],$$

thus, under **(H2)-(H3)**, and by integration on the real component *y*, we obtain

$$\mathbb{E}[H_1^2 | X_1] = O(1),$$

as, for all $j \geq 1$, $\mathbb{E}\big[K_1^j\big] = O\big(\phi_x(h_K)\big)$, then

$$\mathbb{E}[K_1^2 H_1^2] = O\big(\phi_x(h_K)\big).$$

It follows that

$$\left( \frac{1}{n(\mathbb{E}[K_1(x)])^2} \right) Var(K_1 H_1) = O\left( \frac{1}{n\phi_x(h_K)} \right). \tag{5}$$

Now, to deal with the part **T2** from equation (4) in the same way that Masry (1986) developed it, one needs the following decomposition

$$\left|Cov\left(\prod_{i=1}^{u}\delta_{s_i},\prod_{j=1}^{v}\delta_{t_j}\right)\right| = \underbrace{\sum_{i=1}^{u}\sum_{\substack{j=1\\0\le|i-j|\le m_n}}^{v} Cov\big(K_iH_i,K_jH_j\big)}_{\mathbf{P_1}}$$

$$+\underbrace{\sum_{i=1}^{n}\sum_{\substack{j=1\\|i-j|>m_n}}^{n} Cov\big(K_iH_i,K_jH_j\big)}_{\mathbf{P_2}},$$

where $m_n \xrightarrow[n\to\infty]{} \infty$ and from assumptions **(H1)-(H3)-(H6)**, thus for $i \neq j$:

$$\begin{aligned} P_1 &\le nm_n\left(\max_{i\neq j}\left|\mathbb{E}\big(K_iH_i,K_jH_j\big)\right| + (\mathbb{E}[K_1H_1])^2\right) \\ &\le \mathcal{C}nm_n\left(\phi_X^2(h_K) + \big(\phi_x(h_K)\big)^2\right) \\ &\le \mathcal{C}nm_n\left(\phi_X^2(h_K)\right) \end{aligned} \tag{6}$$

and for $\mathbf{P_2}$:

$$\begin{aligned} P_2 &\le (h_K^{-1}\operatorname{Lip}(K) + h_H^{-1}\operatorname{Lip}(H))^2 \sum_{i=1}^{u}\sum_{\substack{j=1\\|i-j|>m_n}}^{v}\lambda_{i,j} \\ &\le \mathcal{C}(h_K^{-1}\operatorname{Lip}(K) + \operatorname{Lip}(H))^2 \sum_{i=1}^{u}\sum_{\substack{j=1\\|i-j|>m_n}}^{v}\lambda_{i,j} \\ &\le \mathcal{C}n(h_K^{-1}\operatorname{Lip}(K) + \operatorname{Lip}(H))^2\lambda_{m_n} \\ &\le \mathcal{C}n\left(h_K^{-1}\operatorname{Lip}(K) + \operatorname{Lip}(H)\right)^2 e^{-\alpha m_n}. \end{aligned} \tag{7}$$

Then, by (6)-(7), we obtain

$$\sum_{j=1,i\neq j}^{n} Cov\big(K_iH_i,K_jH_j\big)$$
$$\le \mathcal{C}n\left(m_n\left(\phi_X^2(h_K)\right) + (h_K^{-1}\operatorname{Lip}(K) + \operatorname{Lip}(H))^2 e^{-\alpha m_n}\right),$$

by choosing

$$m_{n=}\log\left(\frac{(h_K^{-1}\operatorname{Lip}(K) + h_H^{-1}\operatorname{Lip}(H))^2}{\alpha\phi_X^2(h_K)}\right),$$

we obtain

$$\frac{1}{\phi_x(h_K)}\sum_{j=1,i\neq j}^{n} Cov\big(K_iH_i, K_jH_j\big) \to 0, as\ n \to \infty. \tag{8}$$

At last, by collecting the results (4)-(5) and (8), we obtain

$$Var\left(\sum_{i=1}^{n}\delta_i\right) = O\left(\frac{1}{n\phi_x(h_K)}\right),$$

thus, the variables $\delta_i, i = 1, ..., n$ be content with the assumptions **(H1)-(H4)-(H6)**

$$K_n = \frac{C}{n\sqrt{\phi_x(h_K)}}, M_n = \frac{C}{n\phi_x(h_K)} \text{ and } A_n = Var\left(\sum_{i=1}^{n}\delta_i\right).$$

Hence

$$\mathbb{P}\left(\left|\hat{f}_N^x(y) - \mathbb{E}\big[\hat{f}_N^x(y)\big]\right| > \eta\sqrt{\frac{\log(n)}{n\phi_x(h_K)}}\right) = \mathbb{P}\left(\left|\sum_{i=1}^{n}\delta_i\right| > \eta\sqrt{\frac{\log(n)}{n\phi_x(h_K)}}\right)$$

$$\leq exp\left\{-\eta^2\frac{\log(n)}{n\phi_x(h_K)Var\big(\sum_{i=1}^{n}\delta_i\big) + \sqrt[6]{\frac{\log^5(n)}{\big(n\phi_x(h_K)\big)^7}}}\right\}$$

$$\leq exp\left\{-\eta^2\frac{\log(n)}{C + \sqrt[6]{\frac{\log^5(n)}{n\phi_x(h_K)}}}\right\} \tag{9}$$

$$\leq C'exp\left\{-\eta^2\frac{\log(n)}{C + \sqrt[6]{\frac{\log^5(n)}{n\phi_x(h_K)}}}\right\}.$$

Finally, by **(H7)** and for a favourable choice of $\eta$, the Borel-Cantellis lemma allows to finish the proof of **Lemma 4.1.**

**Proof of Corollary 4.1.** We have

$$\left\{\left|\hat{F}_D^x\right| < \frac{1}{2}\right\} \subseteq \left\{\left|\hat{F}_D^x - 1\right| > \frac{1}{2}\right\}.$$

Otherwise

$$\mathbb{P}\left\{\left|\hat{F}_D^x\right| < \frac{1}{2}\right\} \leq \mathbb{P}\left\{\left|\hat{F}_D^x - 1\right| > \frac{1}{2}\right\} \leq \mathbb{P}\left\{\left|\hat{F}_D^x - \mathbb{E}\left[\hat{F}_D^x\right]\right| > \frac{1}{2}\right\},$$

for $\mathbb{E}\left[\hat{F}_D^x\right] = 1$, we apply the result of **Lemma 4.1** to show that

$$\mathbb{P}\left\{\left|\hat{F}_D^x\right| < \frac{1}{2}\right\} \leq \infty.$$

**Proof of Lemma 4.2.** The proof is a direct application of **Lemma 4.1** when the authors replaced $\chi(.,.)$ in equation (1) by

$$\chi(X_i, Y_i) = K(h_K^{-1} d(x, X_i)) - \mathbb{E}[K_1], \forall X_i \in \mathcal{H}.$$

**Proof of Lemma 4.3.** We have

$$\begin{aligned}
\mathbb{E}\left[\hat{F}_N^x(y)\right] - F^x(y) &= \frac{1}{n\mathbb{E}[K_1(x)]} \sum_{i=1}^{n} \mathbb{E}[K_i(x) H_i(y)] - F^X(y) \\
&= \frac{1}{\mathbb{E}[K_1(x)]} \mathbb{E}[K_1(x) H_1(y) - F^X(y)] \\
&= \frac{1}{\mathbb{E}[K_1(x)]} \mathbb{E}(K_1[\mathbb{E}(H_1(y)|X) - F^X(y)]),
\end{aligned}$$

using the stationarity of the observations, the conditioning by the explanatory variable and the usual change of variable $t = \frac{y-u}{h_H}$, we obtain:

$$\begin{aligned}
\mathbb{E}(H(h_H^{-1}(y - Y_i))|X) &= \int_{-\infty}^{+\infty} H(h_H^{-1}(y - Y_i))\, f^X(u) du \\
&= \int_{-\infty}^{+\infty} H'(h_H^{-1}(y - Y_i))\, F^X(u) du \\
&= \int_{-\infty}^{+\infty} H'(t)\, F^X(y - h_H) dt.
\end{aligned}$$

and can deduce that

$$\left|\mathbb{E}(H(h_H^{-1}(y - Y_i))|X) - F^X(y)\right| = \int_{-\infty}^{+\infty} H'(t)\, |F^X(y - h_H t) - F^X(y)| dt.$$

Under **(H3)**: $\forall\, y \in S$,

$$\left|\mathbb{E}(H(h_H^{-1}(y - Y_i))|X) - F^X(y)\right| \le A_X \int_{-\infty}^{+\infty} H'(t)(h_K^{b_1} + |t| h_H^{b_2})\, dt.$$

Hypothesis **(H4)** and **Corollary 4.1** complete the proof of **Lemma 4.3.**

## 6. Conclusion

In this paper, the authors established the consistency properties (with rates) of the conditional density function in a scalar response variable given a random variable taking values in a separable real Hilbert space when the observations are quasi--associated dependent; the pointwise almost complete convergence (with rates) of the kernel estimate of this model was obtained.

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## WŁAŚCIWOŚCI ASYMPTOTYCZNE SZACUNKU ROZKŁADU WARUNKOWEGO DLA POWIĄZANYCH DANYCH FUNKCJONALNYCH

**Streszczenie:** Celem niniejszej pracy jest zbadanie metodą jądra nieparametrycznego oszacowania warunkowej funkcji rozkładu zmiennej odpowiedzi skalarnej przy zmiennej losowej przyjmującej wartości w separowalnej rzeczywistej przestrzeni Hilberta, gdy obserwacje są *quasi*-skojarzone zależne. W pewnych ogólnych warunkach ustala się punktowo prawie zupełną zgodność ze stawkami tego estymatora. Głównym celem jest zbadanie współczynnika zbieżności proponowanego estymatora.

**Słowa kluczowe:** estymacja nieparametryczna, prawdopodobieństwo *small ball*, dane *quasi*-skojarzone.