Wrocław University of Technology

Civil Engineering

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STRUCTURAL DYNAMICS

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Reviewer: Wojciech Glabisz

CONTENTS

INTRODUCTION

This book presents the theory of Single-Degree-Of-Freedom (SDOF) and Multi-Degree-Of-Freedom (MDOF) Systems, which corresponds to the educational program for students of Structural Dynamics at the Faculty of Civil Engineering . Theories of free and forced steady-state vibration of undamped and viscously damped systems are presented and discussed. The equation of motion is derived in two ways: firstly – according to the d'Alembert's Principle, and secondly – according to the Mechanical Energy Balance method with use of Lagrange's equation. Particularly, the comprehensive analysis of single-degree-of-freedom systems is carried out. Multiple degree-of-freedom systems are discussed, including the normal-mode theory of linear elastic structures and Lagrange's equation. The free and forced vibration are analyzed with the use of either the modal transformation approach or the direct method for harmonically excited systems. The finite-number-of-degrees-of-freedom systems are obtained from the ones whose parameters are distributed. This is achieved by lumping parameters for certain regions into equivalent masses, springs, and dampers; each of these is assumed to have only a single function. The plane beams, trusses and foundations subject to the action of dynamic forces are analyzed from the point of view of such discretized systems.

A detailed list of literature is placed at the end of the book. As the authors were once professor Jan Langer's students, the present handbook is mainly based on his book [5].

SYMBOLS AND ACRONYMS

a	acceleration magnitude
a, am q	amplitude
a_{beat}	amplitude of beat
$\mathbf{a}$	vector of amplitudes, vector of acceleration
a_x, a_y, a_z	acceleration components along the coordinate axes x, y and z
A	area
$\mathbf{A}_k, \mathbf{A}_m, \mathbf{A}_d, \mathbf{A}_f, \mathbf{A}_\Delta$	transformation matrices from generalized to local coordinates
b	number of rigid bodies in the 3D system
$\mathbf{B}, \{\mathbf{m}\}$	mass (inertia) matrix, diagonal inertia matrix
c, c_e	damping coefficient, equivalent damping coefficient
c_{cr}	critical damping coefficient
C	constant
$\mathbf{C}, \mathbf{C}_1, \mathbf{C}_2 \{\mathbf{c}\}$	damping matrix, diagonal damping matrix
d	number of degrees of freedom
d_Δ	number of translational degrees of freedom
d_φ	number of rotational degrees of freedom
$\mathbf{D}$	flexibility matrix
$\hat{\mathbf{D}}$	flexibility matrix in expanded base of coordinates
e	eccentricity radius, the number of constraints in the system
E	Young's modulus
E_d^*	energy dissipated during a cycle of vibration
E_k	kinetic energy
E_p	potential energy
E_p^*	potential strain energy
f	frequency
$\mathbf{F}_k$	vector of spring force
$\mathbf{F}_m$	inertia force
$\mathbf{F}_d$	damping force
$\mathbf{F}_r$	vector of static restoring force
$\mathbf{F}_{ex}$	vector of external force
F_T	value of transmitted to foundation force
F_x, F_y, F_z	force components along the coordinate axes x, y and z
F_S, F_C	sinusoidal and cosinusoidal component of force
$\mathbf{F}_S, \mathbf{F}_C$	vectors of sinusoidal and cosinusoidal components of force vector
$\Delta \mathbf{F}(t)$	additional moment of force
G	shear modulus
$\mathbf{G}$	weight, gravity force vector

I	second moment of area or second moment of volume
J	moment of mass inertia of the body
J_ξ, J_O, J_A	moment of mass inertia of the body around the fixed axis ξ, or rotation in plane around the point O or A
k, k_e	spring constant, stiffness, equivalent spring constant, equivalent stiffness
$\mathbf{K}, \{\mathbf{k}\}$	stiffness matrix, diagonal stiffness matrix
$\{\mathbf{k_o}\}$	principle stiffnesses matrix
$\hat{\mathbf{K}}$	stiffness matrix in expanded base of coordinates
$L, \mathbf{L}, \mathbf{L}_O$	moment of momentum value, moment of momentum vector
m, m, M	mass
$\mathbf{M}$	matrix
$\{\mathbf{m_o}\}$	principle masses matrix
m_r	rotating mass
$\mathbf{M}_m$	inertia moment
M_O	moments of force with respect to an axis through O
$\mathbf{M}_{ex}$	vector of external moment
M_S, M_C, M	bending moments,
n_g	degree of geometric indeterminacy
n_{gd}	degree of kinematic (geometric) indeterminacy in a dynamic sense
n_h	degree of static indeterminacy
n_M	number of unknown member forces
n_N	number of independent, non-trivial equilibrium equations
n_φ	number of rotational constraints
n_Δ	number of translational constraints
N_S, N_C, N	axial (normal) forces
$\mathbf{N}_S, \mathbf{N}_C, \mathbf{N}$	forces vector in the set of members
p	number of members in a kinematic chain
$\mathbf{p}$	vector of momentum
$\mathbf{P}$	excitation forces vector in a local base of coordinates
$q, \tilde{q}, \dot{q}, \dot{\tilde{q}}, \ddot{q}, \ddot{\tilde{q}}$	displacement, velocity, acceleration in generalized coordinates
$\mathbf{q}, \tilde{\mathbf{q}}, \dot{\mathbf{q}}, \dot{\tilde{\mathbf{q}}}, \ddot{\mathbf{q}}, \ddot{\tilde{\mathbf{q}}}$	vector of displacements, velocities, accelerations in generalized coordinates
q_m	maximum amplitude of displacement
$q_o, \dot{q}_o$	initial displacement, initial velocity
q_S, q_C	sinusoidal and cosinusoidal component of displacement, constants of integration
$\mathbf{q}_S, \mathbf{q}_C$	sinusoidal and cosinusoidal component of displacements vector, vectors of constants of integration

q_{st}	static deflection
$\hat{\mathbf{q}}$	expanded base of kinematic coordinates vector
$\mathbf{Q}$, Q_i	vector of unknown internal forces, dynamic force
$\hat{\mathbf{Q}}$	dynamic forces vector in expanded base of coordinates
r	number of supporting constraints (links) in a kinematic chain
$r, \dot{r}, \ddot{r}$	displacement, velocity, acceleration in principal coordinates
$\mathbf{r}, \dot{\mathbf{r}}, \ddot{\mathbf{r}}$	displacement, velocity and acceleration vectors in principal coordinates
$\delta\mathbf{r}$	virtual displacement
$\mathbf{R}, \mathbf{R}_{ij}$	matrix, element of the matrix
$\tilde{R}$, $\tilde{r}$	semi-major axis and semi-minor axis of ellipse
$\mathbf{s}, \mathbf{c}$	vectors of sine and cosine functions
S_x, S_y	static moment of mass about the axis x and y
t	time, number of rigid bodies in the 2D system,
T	period
T_d	period of damped vibration
T_S , T_C , T	shear forces
$u, \dot{u}, \ddot{u}$	displacement, velocity, acceleration in local coordinates
$\mathbf{u}, \dot{\mathbf{u}}, \ddot{\mathbf{u}}$	vector of displacements, velocities and accelerations in local coordinates
$u_o, \mathrm{am}\, u$	amplitude of foundation motion
v	velocity magnitude
$\mathbf{v}$, $\mathbf{v}_O$	vector of velocity
V	volume
w	number of truss hinges in a kinematic chain
W	work
δW	virtual work
$\mathbf{w}$	eigenvector
$\mathbf{W}$	modal matrix
$\|\mathbf{w}\|$	norm of vector $\mathbf{w}$
$\left| w_{ji} \right|$	absolute value of w_{ji}, module of w_{ji}
x, y, z	cartesian coordinates
$\mathbf{x}$	kinematic coordinates vector in the dynamic sense
$\mathbf{X}$	hyperstatic forces vector
α	damping ratio (fraction of critical damping)
α_{eq}	equivalent (viscous) damping ratio
γ	non-dimensional damping coefficient
δ	flexibility, compliance
Δ	extensions of the members vector
ε	magnitude of angular acceleration
$\boldsymbol{\varepsilon}$	vector of angular acceleration
η	frequency ratio

ϑ	logarithmic decrement
κ	dimensional parameter – retardation time
λ	root of characteristic equation, eigenvalue
μ	dimensional damping parameter
ν_a	acceleration response factor
$\nu_d, \nu_{d\,\max}, \nu_m$	dynamic magnification factor, maximum value of dynamic amplification factor
ν_r	resonant dynamic magnification factor,
ν_T, ν_T'	response factor, transmissibility
ν_v	velocity response factor
ρ	density of the body or density of the area
$\sigma_S, \sigma_C, \sigma$	values of stresses
φ	initial phase
Φ	Rayleigh dissipation function
$\varphi, \dot{\varphi}, \ddot{\varphi}$	angular displacements, angular velocities, angular accelerations
$\boldsymbol{\varphi}, \dot{\boldsymbol{\varphi}}, \ddot{\boldsymbol{\varphi}}$	vector of angular displacements, velocities and accelerations
$\boldsymbol{\omega}$	vector of angular velocity
ω, Ω	angular frequency, magnitude of angular velocity, angular frequency of excitation
ω_n	natural frequency
ω_d	(angular) frequency of damped vibration, damping frequency
$\omega t + \varphi$	phase
ψ	phase angle delay
$\boldsymbol{\Omega}^2$	spectral matrix
$\mathfrak{I}_o$	moment of unbalance

CHAPTER 1

Basic Ideas

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1. Basic Ideas

1.1. Important Concepts and Definitions

Dynamics

Dynamics is the branch of mechanics that deals with the motion of a system of material particles under the influence of forces, especially ones originating outside of the system

Oscillation

Oscillation is the variation, usually with time, of the magnitude of a quantity with respect to a specified reference. The magnitude alternately becomes smaller and greater than the reference.

Vibration

Vibration is a type of oscillation in which the quantity is a parameter that defines the motion of a mechanical system.

Degrees of Freedom

Degrees of freedom are the ways in which the space configuration of a mechanical system may change, i.e. the independent movements the system can possibly undergo.

Generalized coordinate

Generalized coordinates uniquely define any possible configuration of the system relative to the reference configuration. In this book the generalized coordinates are chosen to be independent of one another.

1.2. Harmonic and Quasi-Harmonic Motion

Harmonic functions are often used to analyze shock and vibration. The following equation describes the displacement pattern followed by a body moving harmonically in time

$$q(t) = a\sin(2\pi f t + \varphi) = a\sin(\omega t + \varphi) \qquad (1.1)$$

where f is the frequency, $\omega = 2\pi f$ is the corresponding angular frequency, $a > 0$ is the amplitude of the displacement, $(\omega t + \varphi)$ is the phase and φ is the initial phase of the harmonic displacement function.

The velocity $\dot{q}(t)$ and acceleration $\ddot{q}(t)$ of the body are found by differentiating the displacement once and twice, respectively:

$$\dot{q}(t) = a(2\pi f)\cos(2\pi f t + \varphi) = a\omega\cos(\omega t + \varphi) \qquad (1.2)$$

$$\ddot{q}(t) = -a(2\pi f)^2 \sin(2\pi f t + \varphi) = -a\omega^2 \sin(\omega t + \varphi) = -\omega^2 q(t) \qquad (1.3)$$

Displacement, velocity and acceleration amplitudes are defined as follows:

$$\text{am}\, q = a, \quad \text{am}\, \dot{q} = \omega\, a, \quad \text{am}\, \ddot{q} = \omega^2 a \qquad (1.4)$$

A harmonic motion described by Eq. (1.1) can be written down in form

$$q(t) = q_S \sin\omega t + q_C \cos\omega t \qquad (1.5)$$

where

$$q_S = a\cos\omega t, \quad q_C = a\sin\omega t \qquad (1.6)$$

or

$$a = \text{am}\, q = \sqrt{q_S^2 + q_C^2}$$
$$\varphi = \text{arctg}(q_C / q_S) \qquad (1.7)$$

The velocity and acceleration can be obtained by differentiating Eq. (1.5) with respect to time twice. After the first differentiation the equation of velocity is achieved in a form equivalent to Eq. (1.2)

$$\dot{q}(t) = q_s\omega\cos\omega t - q_c\omega\sin\omega t \qquad (1.8)$$

After the second differentiation the equation of acceleration is achieved in a form equivalent to Eq. (1.3)

$$\ddot{q}(t) = -q_s\omega^2 \sin\omega t - q_c\omega^2 \cos\omega t \qquad (1.9)$$

The quasi-harmonic motion with amplitude modulation $a(t) > 0$ can be written in form

$$q(t) = a(t)\ \sin(\omega t + \varphi) = q_S(t)\sin\omega t + q_C(t)\cos\omega t \qquad (1.10)$$

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1.3. Characteristics of Harmonic Motion

Amplitude $\quad a = \operatorname{am} q \;$ [m]

The amplitude is the maximum absolute value of the displacement of a body undergoing harmonic motion.

Period $\quad T \,$ [s]

The period of a periodic quantity is the smallest increment of the independent variable for which the function repeats itself i.e. $\underset{T \neq 0}{\exists} \; \underset{t}{\forall} \; f(T + t) = f(t)$.

Frequency $\quad f = 1/T \;$ [1/s=Hz]

The frequency of the function periodic in time is the reciprocal of the period. The unit is a cycle per unit time and must be specified. The unit per cycle per second is called hertz.

Angular frequency (circular frequency) $\quad \omega = 2\pi/T = 2\pi f \;$ [rad/s]

The angular frequency of a periodic quantity, in radians per unit time, is the frequency multiplied by 2π.

Phase angle $\quad (\omega t + \varphi) \;$ [rad]

The phase of a periodic quantity, for a periodic value of the independent variable, is the fractional part of a period through which the independent variable has advanced, measured from an arbitrary reference.

Initial phase angle $\quad \varphi = (\omega t + \varphi)\big|_{t=0} \;$ [rad]

The phase of a periodic quantity, for a periodic value of the independent variable, is the fractional part of a period through which the independent variable has advanced, measured from an arbitrary reference, in time point equal to zero.

1.4. Superposition of Harmonic Moves

A point which undergoes simultaneous excitation from two independent sources exhibits vibration that is a sum of two vibrations. This phenomenon is referred to as superposition of vibration. While superposition may be applied to vibration whose function in time takes on any shape, only the case of harmonic vibration will be considered here. There are two possibilities that should be taken into account:

- superposition of parallel vibration - both movements occur in the same direction. This situation may result in the phenomenon of beats.
- superposition of perpendicular vibration – the movements occur perpendicularly to one another. Under the influence of the resultant vibration the point moves along a complicated trajectory. The least complicated of the curves that reflect those trajectories graphically are called Lissajous curves.

1.4.1. Superposition of Synchronous Moves

Let us consider a movement of the point described by the function $q(t)$ which is a combination of the collinear harmonic movements $q_j(t)$, each of which is described by

$$q_j(t) = a_j \sin(\omega_j t + \varphi_j) \tag{1.11}$$

where $\omega_j = \omega = const$, $\varphi_j = \varphi = const$. Such movements are called synchronous. Then

$$q(t) = \left(\sum_j a_j \right) \sin(\omega_j t + \varphi_j) = a \sin(\omega t + \varphi) \tag{1.12}$$

The amplitude of resultant movement is a sum of component amplitudes, i.e.

$$a = \sum_j a_j . \tag{1.13}$$

Conclusion:

The resultant movement is harmonic and synchronous with component movements.

1.4.2. Superposition of Isochronous Moves

Let us consider a movement of the point described by the function $q_j(t)$, Eq. (1.11), where $\omega_j = \omega = const$, but the phase angles are different, i.e. $\varphi_j \neq \varphi_i$. Such movements are called isochronous. Then

$$q_j(t) = a_j \sin(\omega t + \varphi_j) = a_j \cos \varphi_j \sin \omega t + a_j \sin \varphi_j \cos \omega t \tag{1.14}$$

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The resultant movement can be expressed by formula

$$q(t) = \left(\sum_j a_j \cos \varphi_j \right) \sin \omega t + \left(\sum_j a_j \sin \varphi_j \right) \cos \omega t \qquad (1.15)$$

By introducing new vectors, which can be defined

$$\mathbf{a} = [a_1 \quad a_2 \quad a_3 \quad \cdots]^{\mathrm{T}} \qquad (1.16)$$

$$\mathbf{c} = [\cos \varphi_1 \quad \cos \varphi_2 \quad \cos \varphi_3 \quad \cdots]^{\mathrm{T}} \qquad (1.17)$$

$$\mathbf{s} = [\sin \varphi_1 \quad \sin \varphi_2 \quad \sin \varphi_3 \quad \cdots]^{\mathrm{T}} \qquad (1.18)$$

the formula Eq. (1.15) can be written in form

$$q(t) = \mathbf{a}^{\mathrm{T}} \mathbf{c} \, \sin \omega t + \mathbf{a}^{\mathrm{T}} \mathbf{s} \sin \omega t = a \sin(\omega t + \varphi) \qquad (1.19)$$

where

$$\varphi = \operatorname{arctg} \left(\frac{\mathbf{a}^{\mathrm{T}} \mathbf{s}}{\mathbf{a}^{\mathrm{T}} \mathbf{c}} \right) \qquad (1.20)$$

and

$$a^2 = \mathbf{a}^{\mathrm{T}} \mathbf{c} \mathbf{c}^{\mathrm{T}} \mathbf{a} + \mathbf{a}^{\mathrm{T}} \mathbf{s} \mathbf{s}^{\mathrm{T}} \mathbf{a} = \mathbf{a}^{\mathrm{T}} \left(\mathbf{c} \mathbf{c}^{\mathrm{T}} + \mathbf{s} \mathbf{s}^{\mathrm{T}} \right) \mathbf{a} = \mathbf{a}^{\mathrm{T}} \mathbf{R} \, \mathbf{a} \qquad (1.21)$$

The elements of matrix $\mathbf{R}$ can be calculated according to formulas

$$\mathbf{R}_{ij} = \cos \varphi_i \cos \varphi_j + \sin \varphi_i \sin \varphi_j = \cos \left| \varphi_i - \varphi_j \right| \qquad (1.22)$$

$$\mathbf{R}_{ij} = \mathbf{R}_{ji}, \qquad \mathbf{R}_{ii} = 1 \qquad (1.23)$$

$$a = \sqrt{\mathbf{a}^{\mathrm{T}} \mathbf{c} \mathbf{c}^{\mathrm{T}} \mathbf{a} + \mathbf{a}^{\mathrm{T}} \mathbf{s} \mathbf{s}^{\mathrm{T}} \mathbf{a}} = \sqrt{\mathbf{a}^{\mathrm{T}} \left(\mathbf{c} \mathbf{c}^{\mathrm{T}} + \mathbf{s} \mathbf{s}^{\mathrm{T}} \right) \mathbf{a}} = \sqrt{\mathbf{a}^{\mathrm{T}} \mathbf{R} \, \mathbf{a}} \quad . \qquad (1.24)$$

Conclusion:

- **The resultant movement is harmonic with an angular frequency ω.**
- **The amplitude of this movement can be calculated from formula**

$$a = \sqrt{\mathbf{a}^{\mathrm{T}} \mathbf{R} \, \mathbf{a}} > 0$$

Illustrative Example 1.1

In the case of two movements

$$\mathbf{R} = \begin{bmatrix} 1 & \cos|\varphi_2 - \varphi_1| \\ \cos|\varphi_1 - \varphi_2| & 1 \end{bmatrix} \tag{1.25}$$

thus

$$a = \sqrt{a_1^2 + a_2^2 + 2a_1a_2\cos|\varphi_1 - \varphi_2|} \tag{1.26}$$

For example, when

$$\varphi_1 - \varphi_2 = 0 \qquad \text{then} \qquad a = a_1 + a_2$$

$$|\varphi_1 - \varphi_2| = \frac{\pi}{2} \qquad \text{then} \qquad a = \sqrt{a_1^2 + a_2^2} \tag{1.27}$$

$$|\varphi_1 - \varphi_2| = \pi \qquad \text{then} \qquad a = |a_1 - a_2|$$

1.4.3. Superposition of Asynchronous Moves

Let us consider a movement of the point as a superposition of movements described by Eq. (1.11), where angular frequencies ω_j are different – asynchronous moves . The resultant movement $q(t)$ is not a harmonic one. This movement is periodic if the proportion of the angular frequencies of component movements is a rational number, i.e.

$$\omega_1 : \omega_2 : \omega_3 : \cdots = n_1 : n_2 : n_3 : \cdots \tag{1.28}$$

where n_j are relatively prime natural numbers (i.e. their set does not have a common divisor). Then, it can be written $\omega_j/n_j = const$, and from here

$$n_j T_j = T = const \tag{1.29}$$

Period T of the resultant movement is the lowest common multiple of the component movements periods. If n_j are small numbers, period T is comparable to periods of component movements. In the opposite situation, period T can be many times greater than the periods of component movements. If any of the n_j is not a rational number (for example $\omega_1 = 1$, $\omega_2 = \sqrt{2}$), the period achieves infinity. The maximum absolute value of resultant movement is described by formula

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$$\max_t \max |q(t)| \leq \sum_j a_j \qquad (1.30)$$

In the case of superposition of two harmonic movements for which $\omega_1 \approx \omega_2$ and $a_1 = a_2$ or $a_1 \approx a_2$, the superposition of these asynchronous moves leads to a phenomenon which is called beats.

Conclusion:

- **Generally, the resultant movement is not a harmonic one.**

- **This movement is periodic only if the proportion of the angular frequencies of component movements is a rational number**

Illustrative Example 1.2
Superposition of Parallel Vibrations – Beats

Let us consider a periodic vibration which appears as a result of a superposition of two simple harmonic quantities (vibrations) occurring in the same direction but with different frequencies, f_1 and f_2. The amplitudes of component functions are the same $a_1 = a_2 = a$ or almost the same $a_1 \approx a_2$.

As a result of the superposition of these parallel vibrations, the phenomenon of beats appears, Fig. 1.1. A periodic increase and decrease of amplitude at the beat frequency $(f_1 - f_2)$ can be observed.
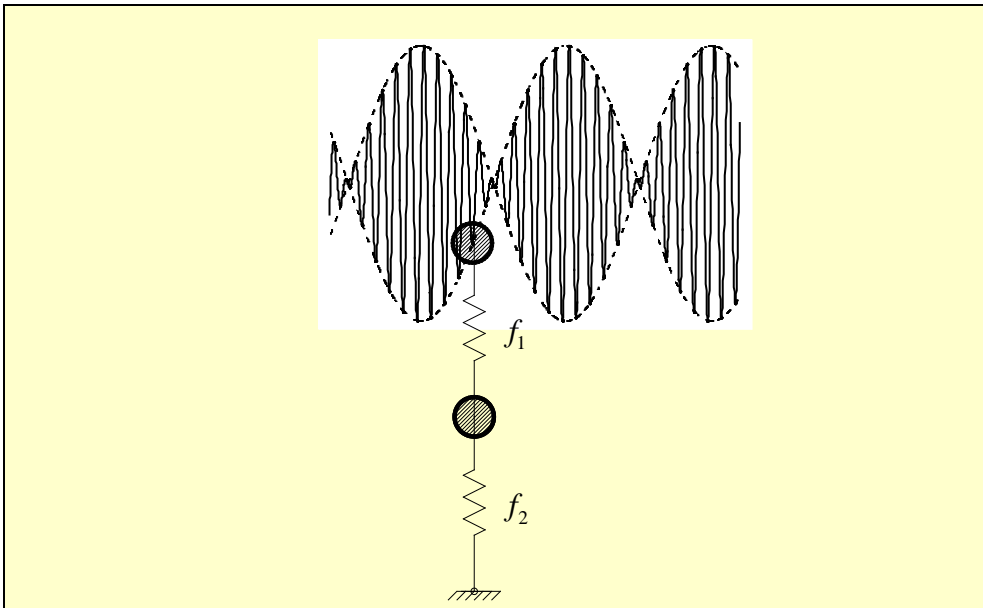
Fig. 1.1 Superposition of parallel vibrations – Beats

One can assume, according to Eq. (1.11), that the components of vibration have a form

$$q_1(t) = a\sin(\omega_1 t + \varphi_1)$$ (1.31)

$$q_2(t) = a\sin(\omega_2 t + \varphi_2)$$ (1.32)

Since the functions Eqs. (1.31) and (1.32) have different frequencies, in any given frame of reference such points in time exist, in which both functions are in phase. If we then assume one of these points in time to be the beginning of a new frame of reference, we can simplify the mathematical notation of the phenomenon without changing the degree of the generality of notation. In the end, the following forms of functions describing movement are assumed:

$$q_1(t) = a\sin\omega_1 t$$ (1.33)

$$q_2(t) = a\sin\omega_2 t$$ (1.34)

The resultant move can be calculated by summing Eq.(1.33) and (1.34), namely

$$q(t) = q_1(t) + q_2(t) = a\sin\omega_1 t + a\sin\omega_2 t$$ (1.35)

Since

$$\sin\alpha + \sin\beta = 2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$$ (1.36)

therefore formula Eq. (1.35) can be written in form

$$\begin{aligned} q(t) &= \left[2a\cos\left(\frac{\omega_1-\omega_2}{2}t\right)\right]\sin\left(\frac{\omega_1+\omega_2}{2}t\right) = \\ &= \left[2a\cos\left(2\pi\frac{f_1-f_2}{2}t\right)\right]\sin\left(2\pi\frac{f_1+f_2}{2}t\right) = \\ &= \left[2a\cos(2\pi f_{am}t)\right]\sin\left(2\pi\bar{f}\,t\right) \end{aligned}$$ (1.37)

Resultant movement (resultant vibrations) can be thought of as the movement with frequency

$$\bar{f} = \frac{f_1+f_2}{2}$$ (1.38)

which is an arithmetic mean of two movement frequencies. The amplitude describes the formula in square bracket in Eq. (1.37). It can be seen that amplitude of the resultant movement $q(t)$ changes in time with frequency

$$f_{am} = \frac{f_1 - f_2}{2} \qquad (1.39)$$

Since frequencies $f_1 \approx f_2$ are almost the same, the value of frequency f_{am}, Eq. (1.39), is small and the amplitude (expression in square brackets in formula Eq. (1.37)) changes slowly. This phenomenon is called amplitude modulation. The amplitude of the beats is up to two times greater than the amplitudes of movement components, and its maximum appears when

$$\cos\left(2\pi \frac{f_1 - f_2}{2} t\right) = \pm 1 \qquad (1.40)$$

Since these values occur twice in one period, the frequency of beats is equal to doubled frequency f_{am} i.e. is equal to

$$f_{beat} = |f_1 - f_2| \qquad (1.41)$$

Fig. 1.2 shows a graph of the beats phenomena when the amplitudes of component functions are the same $a_1 = a_2 = a$ or almost the same $a_1 \approx a_2$.

Fig. 1.3 shows a graph of the beats phenomena when the amplitudes of component functions are not the same $a_1 \neq a_2$ and amplitude of beat is now described by expression

$$a_{beat} = \sqrt{a_1^2 + a_2^2 + 2a_1 a_2 \cos(\omega_2 - \omega_1)t} \qquad (1.42)$$
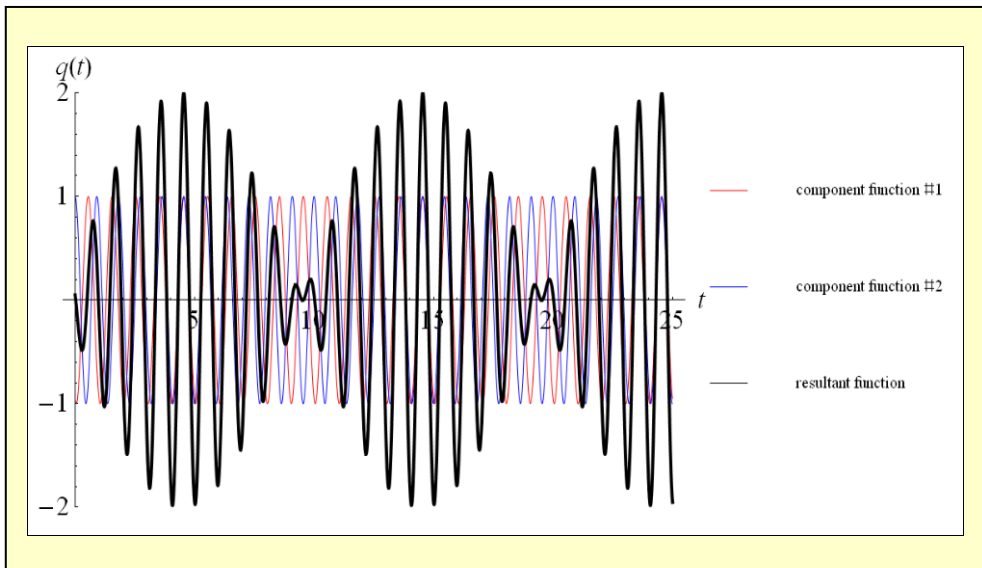
Fig. 1.2 Beats – the amplitudes of component functions are the same $a_1 = a_2 = a$
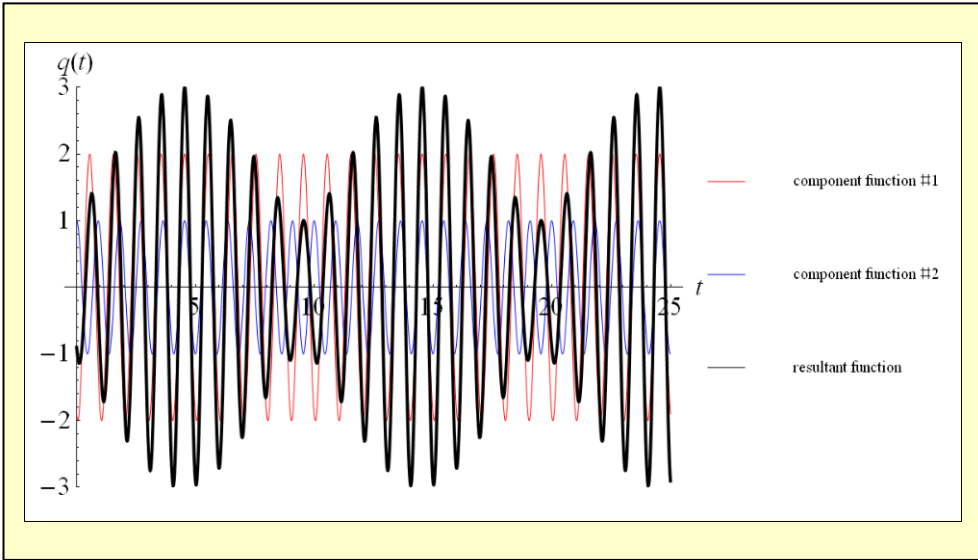
Fig. 1.3 Beats – the amplitudes of component functions are not the same $a_1 \neq a_2$

Illustrative Example 1.3
Superposition of Perpendicular Vibrations – Lissajous Curves

Let us consider a periodic vibration which appears as a result of a superposition of two simple harmonic quantities (vibrations) occurring in directions perpendicular to one another, and with different frequencies, $f_1 = f_x$ and $f_2 = f_y$, Fig. 1.4.
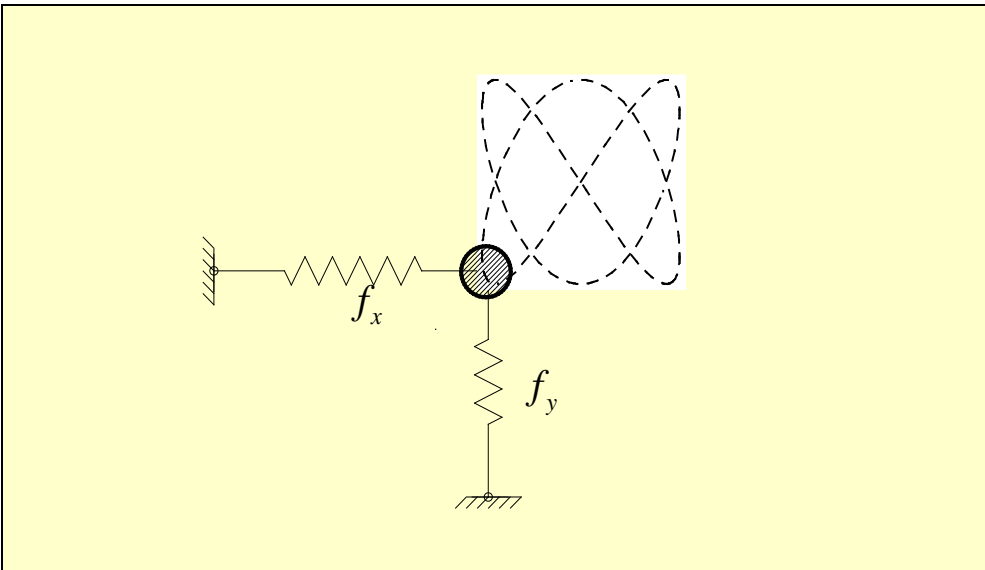

Fig. 1.4 Superposition of perpendicular vibrations – Lissajous curve

Such a situation is referred to as superposition of perpendicular vibration. The trajectories of the resultant motion are often very complex curves. The least complicated of these are called Lissajous curves.

In mathematics, a Lissajous curve is the graph of a system of two parametric equations

$$x(t) = a_x \sin(\omega_x t)$$

(1.43)

$$y(t) = a_y \sin(\omega_y t + \varphi)$$

(1.44)

which describe complex harmonic motion. The equation of the point's trajectory can be obtained from equations of vibration by eliminating the parameter t (time).

The trajectory of the point is a plane curve, whose shape depends on the ratio of both angular frequencies ω_x/ω_y and on the phase shift φ between the vibrations. The shape of the figure is highly sensitive to the ratio ω_x/ω_y, Fig. 1.5.

The problem of establishing the trajectory of the point can be divided into two main cases: the frequencies are the same or they are not the same.

<u>Frequencies are the same</u> $\omega_x = \omega_y = \omega$

From the Eq. (1.43) it can be obtained

$$\sin(\omega t) = \frac{x}{a_x}$$

(1.45)

then

$$\cos(\omega t) = \pm\sqrt{1 - \frac{x^2}{a_x^2}}.$$

(1.46)

After taking into consideration Eq. (1.44) and the trigonometric formula

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

(1.47)

the Eq. (1.44) can, with the use of Eqs. (1.45), (1.46) and (1.47), be written in the form

$$\frac{y}{a_y} = \frac{x}{a_x}\cos\varphi \pm \sqrt{1 - \frac{x^2}{a_x^2}}\sin\varphi$$

(1.48)

After squaring and transforming the Eq. (1.48), one can write finally

$$\frac{x^2}{a_x^2} + \frac{y^2}{a_y^2} - \frac{2xy}{a_x a_y}\cos\varphi = \sin^2\varphi$$

(1.49)

This is an equation of an ellipse tilted under the angle φ to the axis of a frame of reference.

20

Special cases of an ellipse (the first column in Fig. 1.5):

- For $\varphi = 0$ the trajectory is described by the equation

$$y = \frac{a_y}{a_x} x \qquad (1.50)$$

It means that the Lissajous curve is a line segment.

- For $a_x \neq a_y$, $\varphi = \pm \pi/2$ the trajectory is described by the equation

$$\frac{x^2}{a_x^2} + \frac{y^2}{a_y^2} = 1 \qquad (1.51)$$

It means that the Lissajous curve is an ellipse whose axes are in accordance with the axes of a frame of reference.

- For $a_x = a_y = a$, $\varphi = \pi/2$ the trajectory is described by the equation

$$x^2 + y^2 = a^2 \qquad (1.52)$$

It means that the Lissajous curve becomes a circle.
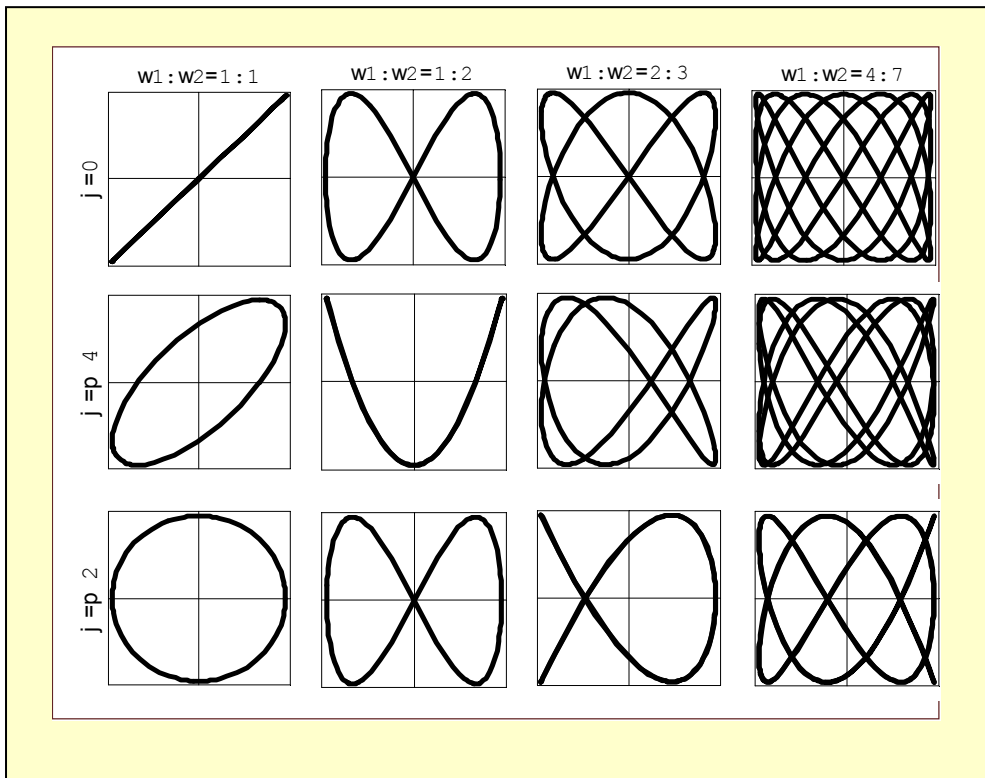
Fig. 1.5 Lissajous figures – periodic vibration

Frequencies are not the same $\omega_x \neq \omega_y$

- For ratio $\omega_x/\omega_y = 1/2$, $\varphi = \pi/2$ the figure is a parabola.
- For ratio ω_x/ω_y which is rational, these curves are closed. It means that resultant movement is periodic, though often very complex.
- Lissajous figures where $\omega_x = 1$, $\omega_y = N$ (N is a natural number) and

$$\varphi = \frac{N-1}{N}\frac{\pi}{2} \qquad (1.53)$$

are Chebyshev polynomials of the first kind of degree N.

- For other ratios ω_x/ω_y the curves are more complicated. The more complex curves are often similar in appearance to three-dimensional knots. In this sense, Lissajous curves are the projections of these knots onto a plane.
- Under continuous change of phase φ the Lissajous curves change shape, giving the impression of dancing in three-dimensional space. In such cases the movement may be very complex – the trajectory may not even be a closed curve, which means that the movement is not periodic, Fig. 1.6.
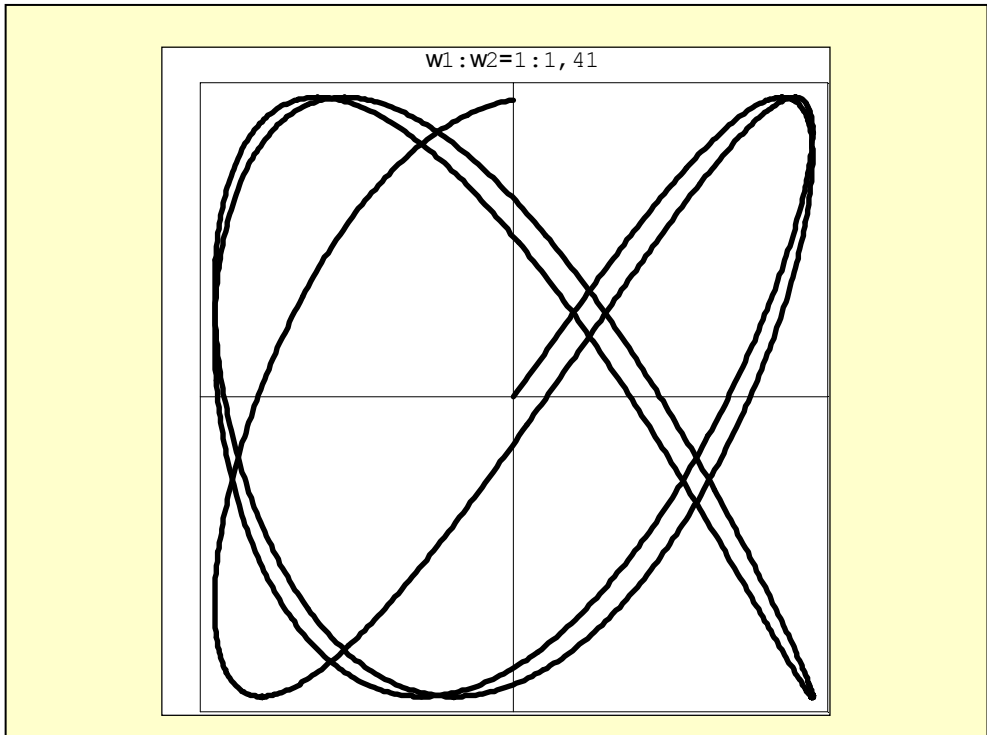
Fig. 1.6 Lissajous figures – non-periodic vibration

1.5. Translational and Rotational Motion

In this chapter the most important definitions of physical quantities describing motion are presented. The same physical quantity can be defined in different ways. The definitions are generally formulated with the use of [1,2,3].

An inertial frame of reference
- An inertial frame of reference is one in which Newton's First Law of Motion is true.
- A frame of reference is inertial if it remains at rest or moves uniformly in a straight line, neither rotating nor accelerating in relation to the stars.

Displacement vector q [m]
- A vector quantity that specifies the change of position of a body or particle. It is usually measured from the mean position or position of rest. It is a vector quantity having direction as well as magnitude.
- A linear distance from the initial to the final position of an object moved from one place to another (regardless of the length of path followed).
- A distance vector of an oscillating particle from its equilibrium position.

Displacement q [m] – the magnitude of the displacement

Velocity vector v, $\dot{\mathbf{q}}$ [m/s]
- A vector quantity that specifies the time rate of change of displacement vector with respect to a reference frame. If the reference frame is not inertial, the velocity is often designated "relative velocity".
- A time rate of change of position of a body; it is a vector quantity having direction as well as magnitude.
- A first derivative of the displacement vector **q** with respect to time $\mathbf{v} = d\mathbf{q}/dt = \dot{\mathbf{q}}$.

Speed or velocity v, $\dot{q}$ [m/s] – the magnitude of velocity

---

Acceleration vector   a, $\ddot{\mathbf{q}}$ [m/s^2]
- A vector quantity that specifies the time rate of velocity of a body; it is a vector quantity having direction as well as magnitude.
- A first derivative of the velocity vector **v** with respect to time $\mathbf{a} = d\mathbf{v}/dt = \dot{\mathbf{v}}$.
- A second derivative of displacement vector **q** with respect to time $\mathbf{a} = d^2\mathbf{q}/dt^2 = \ddot{\mathbf{q}}$.

Acceleration magnitude a or $\ddot{q}$ [m/s^2] – the magnitude of acceleration

Angular displacement vector $\boldsymbol{\varphi}$, $\mathbf{q}$ [rad]

- A direction of the displacement vector is the same as the direction of the axis of rotation and perpendicular to the plane of the angle. The length (magnitude) of the vector is equal to the angle, and the sense is determined by the "right hand rule".
- An angular distance of an oscillating body from its equilibrium position.

Angular displacement or angle φ, q [rad]

- A magnitude of the angular displacement vector $\boldsymbol{\varphi}$.
- The angle is the measure of the inclination of one line or surface with respect to another, equal to the amount that one would have to be turned in order to have the same inclination as the other.

Angular velocity vector $\boldsymbol{\omega}$, $\dot{\boldsymbol{\varphi}}$, $\dot{\mathbf{q}}$ [rad/s]

- A time rate of change of position of angular displacement; it is a vector quantity having direction as well as magnitude.
- A first derivative of the angular displacement vector $\boldsymbol{\varphi}$ with respect to time $\boldsymbol{\omega} = d\boldsymbol{\varphi}/dt$

Angular velocity ω [rad/s] – the magnitude of angular velocity

Angular acceleration vector $\boldsymbol{\varepsilon}$, $\dot{\boldsymbol{\omega}}$, $\ddot{\boldsymbol{\varphi}}$, $\ddot{\mathbf{q}}$ [rad/s^2]

- A time rate of change of angular velocity; it is a vector quantity having direction as well as magnitude.
- A first derivative of the angular velocity vector $\boldsymbol{\omega}$ with respect to time $\boldsymbol{\varepsilon} = d\boldsymbol{\omega}/dt = \dot{\boldsymbol{\omega}}$.
- A second derivative of angular displacement vector $\boldsymbol{\varphi}$ with respect to time $\boldsymbol{\varepsilon} = d^2\boldsymbol{\varphi}/dt^2 = \ddot{\boldsymbol{\varphi}}$.

Angular acceleration ε, $\dot{\omega}$, $\ddot{\varphi}$, $\ddot{q}$ [rad/s^2] – the magnitude of angular

1.6. Newton's Laws of Motion

Newton's Laws of Motion (called Newton's First, Second and Third Laws) are fundamental to classical mechanics. These laws have been proven to be valid for all mechanical problems in an inertial reference frame. They are directly applicable to bodies idealized as particles, that is, bodies assumed to possess mass but no volume. In elementary mechanics, however, it is proven that Newton's Laws of Motion can also be directly applied to bodies of finite dimensions.

1.6.1. First Law

In an inertial frame of reference a particle not subjected to external forces remains at rest or moves with constant speed in a straight line.

1.6.2. Second Law

The objective of study in this chapter is to describe motion (displacement $\mathbf{q}$ or velocity $\mathbf{v} = d\mathbf{q}/dt = \dot{\mathbf{q}}$) of the mass particle m at any time t, for a given set of initial conditions at time $t = 0$. The analytical relation between the displacement $\mathbf{q}$ and time t in an inertial frame of reference is given by Newton's Second Law of Motion

$$\mathbf{F}_{ex} = \frac{d\mathbf{p}}{dt} = \frac{d(m\mathbf{v})}{dt} = m\frac{d\mathbf{v}}{dt} + \mathbf{v}\frac{dm}{dt} = m\mathbf{a} + \mathbf{v}\frac{dm}{dt} \tag{1.54}$$

where $\mathbf{p}$ is the vector momentum of the particle (as a product of its mass m and velocity $\mathbf{v}$, $\mathbf{p} = m\mathbf{v}$), $\mathbf{F}_{ex}$ is the resultant vector (net force) of external forces acting on the particle with mass m and $\mathbf{a}$ is its acceleration vector defined as the first derivative of the velocity vector with respect to time or the second derivative of displacement with respect to time i.e. $\mathbf{a} = d^2\mathbf{q}/dt^2 = \ddot{\mathbf{q}}$.

If the mass particle is not changing in time ($m = const$) the second term on the right side of Eq. (1.54) is equal to zero and Newton's Second Law of Motion can be written in classical form

$$\mathbf{F}_{ex} = m\mathbf{a} \tag{1.55}$$

Eq. (1.55) is a vector relation and as such it can be written in equivalent form in terms of its components along the coordinate axes x, y and z, namely,

$$\sum F_x = m\,a_x \tag{1.56}$$

$$\sum F_y = m\,a_y \tag{1.57}$$

$$\sum F_z = m\,a_z \tag{1.58}$$

The general motion of a rigid body is described by two vector equations:
the first one describes the translational motion with the velocity of its mass center O

$$\frac{d}{dt}(m\mathbf{v}_O) = \mathbf{F}_{ex}$$ (1.59)

the second one describes the rotational motion of the body around its mass center O

$$\frac{d\mathbf{L}_O}{dt} = \mathbf{M}_{ex}$$ (1.60)

In above equations:

m is the mass of the rigid body

$\mathbf{v}_O$ is the velocity of center of mass of the body ,

$\mathbf{F}_{ex}$ is the resultant of external forces (net force) acting on the body,

$\mathbf{M}_{ex}$ is the resultant moment vector of external forces acting on the body with respect to the point O, which is the center of mass of the body,

$\mathbf{L}_O$ is the resultant moment of momentum vector with respect to center of mass of the body.

       Although this last equation expressed in its scalar components is quite complicated, it is not usually needed in structural dynamics.
       Let us consider the special case of a movement – the rigid body rotation around a fixed axis. In this case, Newton's Second Law of Motion can now be written in form

$$\mathbf{M}_{ex} = \frac{d\mathbf{L}}{dt} = \frac{d(J\boldsymbol{\omega})}{dt} = J\frac{d\boldsymbol{\omega}}{dt} + \boldsymbol{\omega}\frac{dJ}{dt} = J\boldsymbol{\varepsilon} + \boldsymbol{\omega}\frac{dJ}{dt}$$ (1.61)

where:

$\boldsymbol{\omega}$ is the angular velocity vector ($\boldsymbol{\omega} = d\boldsymbol{\alpha}/dt = \dot{\boldsymbol{\alpha}}$),

$\boldsymbol{\alpha}$ is the angular displacement vector (rotation around the fixed axis), whose direction is the same as the axis,

J is the moment of mass inertia of the body around the fixed axis of rotation,

$\mathbf{M}_{ex}$ is the resultant moment vector of external forces determined with respect to the same axis of rotation,

L is the resultant moment of momentum vector ($L = J\omega$),

$\boldsymbol{\varepsilon}$ is the angular acceleration vector defined as $\boldsymbol{\varepsilon} = d^2\boldsymbol{\alpha}/dt^2 = \ddot{\boldsymbol{\alpha}}$.

If the mass moment of inertia of the body is not changing in time ($J = const$) the second term on the right side of Eq. (1.61) is equal to zero and Newton's Second Law of Motion for rigid body rotation around a fixed axis can be written in classical form

$$\mathbf{M}_{ex} = J\boldsymbol{\varepsilon}$$ (1.62)

or

$$\sum M_{\xi} = J_{\xi}\,\varepsilon_{\xi}$$ (1.63)

where

ε_{ξ} is the angular acceleration vector around the fixed axis ξ,

J_{ξ} is the moment of mass inertia of the body with respect to the fixed axis ξ,

$\sum M_{\xi}$ is the sum of the moments of all the external forces acting on the body with respect to the fixed axis ξ.

Newton's Second Law of Motion yields the following equations for plane motion of a rigid body:

$$\sum F_x = m\,a_x$$ (1.64)

$$\sum F_y = m\,a_y$$ (1.65)

$$\sum M_O = J_O\,\varepsilon$$ (1.66)

In above equations:

a_x, a_y are the acceleration component, along the x and y axes, of the point O (center of mass of the body);

ε is the angular acceleration around the mass centre point O,

J_O is the mass moment of inertia of the body with respect to an axis through the mass center O, perpendicular to the x–y plane,

$\sum M_O$ is the sum of the moments of all the external forces acting on the body with respect to an axis through O, perpendicular to the *x-y* plane.

1.6.3.   Third Law (Law of Action and Reaction)

If in an inertial frame of reference two particles interact, the force $\mathbf{F}$ exerted by the first particle on the second particle (called the action force) is equal in magnitude and opposite in direction to the force ($-\mathbf{F}$) exerted by the second particle on the first particle (called the reaction force). The action and reaction forces both act along the same line but in the opposite directions. Each of them acts on a different particle, however.

1.7. Elementary Parts of Vibratory Systems

A real vibratory system is able to store both kinds of mechanical energy, potential and kinetic, and to dissipate them. This is achieved with the means of springs for storing potential energy, mass or inertia for storing kinetic energy, and damping for dissipating energy. In an undamped system, the vibration is characterized by mechanical energy only, i.e. it is a continual shift from one to the other kind of mechanical energy (from potential to kinetic and back again). In a damped system, part of the energy is dissipated, i.e. is transferred to non-mechanical forms of energy (for instance to heat). A damped system, on the other hand, requires external sources of energy for the vibration to be maintained, as each cycle of oscillation causes some energy to be lost from the system. While it is possible for one structure to perform all the above functions (storing both kinds of energy and dissipating it), only discrete and lumped parameter systems composed of ideal springs, masses, and dampers will be considered here. In such systems, each element performs one function only. In translational motion, displacements are defined as linear distances; in rotational motion, displacements are defined as angular motions.

1.7.1. Springs

Springs are shown: in Fig. 1.7 linear (translational motion) and in Fig. 1.8 rotational (rotational motion).
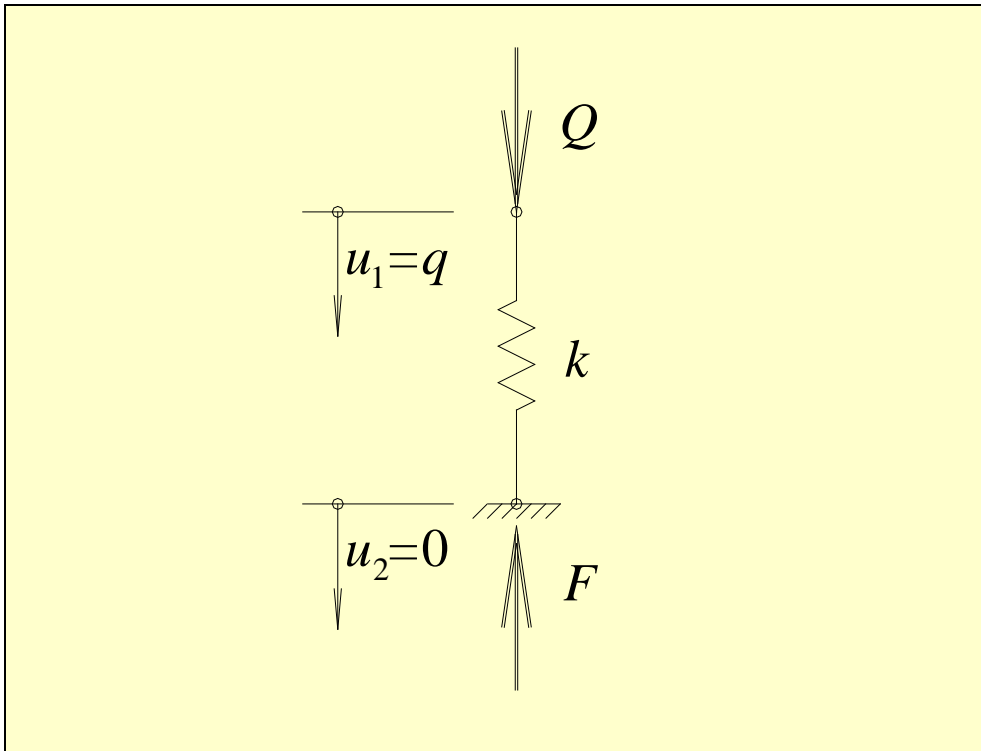
Fig. 1.7 Linear spring.
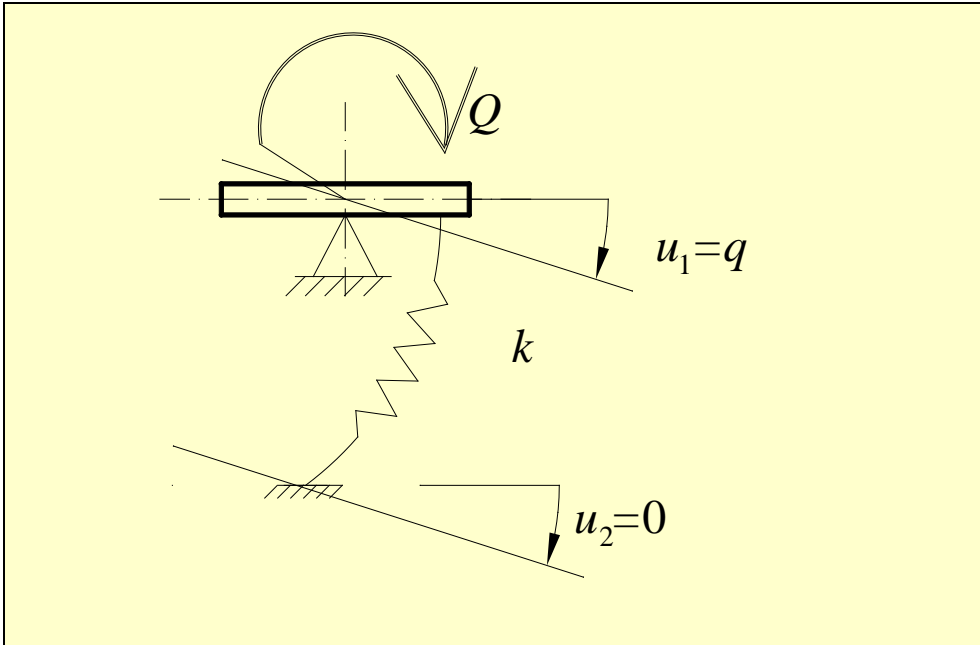
Fig. 1.8 Rotational spring.

1.7.1.1. Spring Law

In the linear spring shown in Fig. 1.7 and the rotational one shown in Fig. 1.8 the change in the length of the spring is proportional to the external generalized force $\mathbf{Q}$ acting on the spring (or moment of force for rotational spring). According to Hooke's Law of Elasticity the restoring force (elastic reaction, spring force) $\mathbf{F}_k = -\mathbf{Q}$ exerted by the spring material is described by the formula

$$\mathbf{F}_k = -k \cdot \Delta\mathbf{u} = -k\,(\mathbf{u}_1 - \mathbf{u}_2) \tag{1.67}$$

where the minus sign means that the force vector is oppositely directed to the displacement vector $\Delta\mathbf{u} = \mathbf{u}_1 - \mathbf{u}_2$ (the difference of the displacement vectors of both ends of the spring along the spring axis, see also Chapter 1.9.1). If displacement $\mathbf{u}_1 = \mathbf{q}$ and $\mathbf{u}_2 = \mathbf{0}$, the Spring Law can be written in form

$$\mathbf{F}_k = -k\,\mathbf{q} \tag{1.68}$$

There are negative signs on the right hand sides of the Eqs. (1.67), (1.68) because the restoring force always acts in the opposite direction to the displacement.
Spring constant or stiffness $k = F/q$ is the constant of proportionality in Eq. (1.68).
Flexibility or compliance $\delta = q/F$ is the inverse of stiffness

$$\delta = \frac{1}{k} \tag{1.69}$$

1.7.1.2. Potential Energy

Potential energy is the work (dot product of force and displacement) that must be performed for a body to be moved from a point where the energy of the body is equal to zero – that is, from a point where the force also equals zero. Thus, potential energy is also a measure of the work the body itself is able to perform thanks to the change in its position.

For one spring, the work can be calculate from formula

$$W = \frac{1}{2} F \cdot \Delta u = \frac{1}{2} k \,(\Delta u)^2 = \Delta E_p \qquad (1.70)$$

where $\Delta u = |\Delta \mathbf{u}| = |\mathbf{u}_1 - \mathbf{u}_2|$.

If $\mathbf{u}_1 = \mathbf{q}$ and $\mathbf{u}_2 = \mathbf{0}$ then $\Delta u = q$ and the work in Eq. (1.70) is equal to the potential energy which can be written in form

$$E_p = \frac{1}{2} k \, q^2 \qquad (1.71)$$

1.7.2. Rigid Mass Body
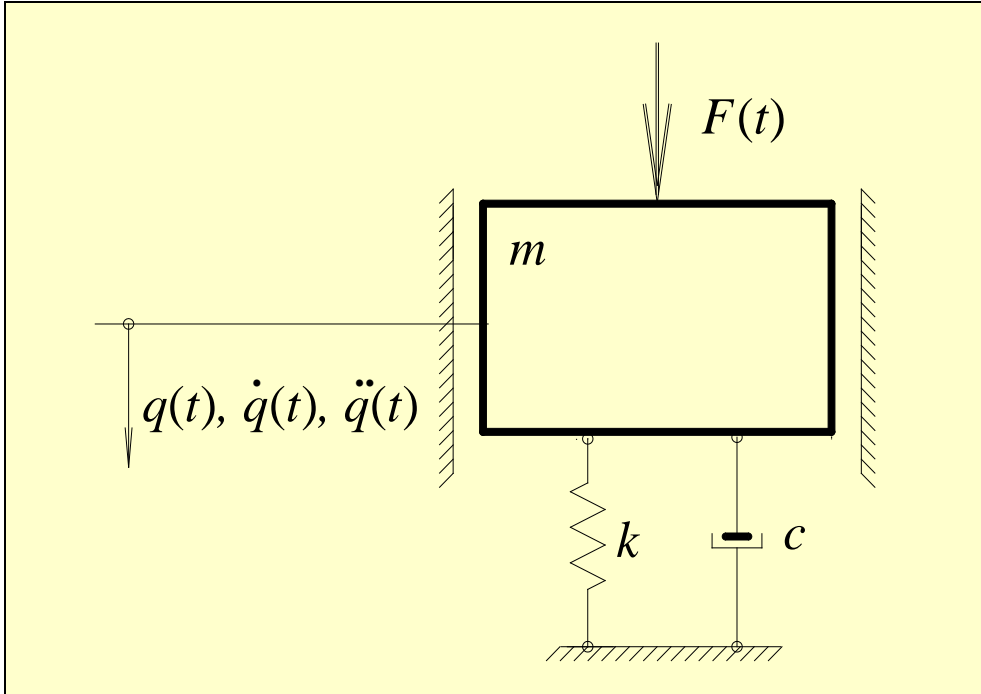
Fig. 1.9 Rigid Mass Body.

1.7.2.1. Mass and Inertia Law

The mass m shown in Fig. 1.9 is a rigid body whose translational acceleration $\ddot{\mathbf{q}}$, according to Newton's Second Law of Motion in an inertial reference frame, is proportional to the resultant force (net force) of all forces acting on the mass

$$\mathbf{F} = m\ddot{\mathbf{q}} \qquad (1.72)$$

The mass m shown in Fig. 1.9 is a rigid body whose translational acceleration $\ddot{\mathbf{q}}$, in a non-inertial reference frame, is proportional to the inertial force according to the formula

$$\mathbf{F}_m = -m\ddot{\mathbf{q}} \qquad (1.73)$$

"Inertial force is a fictitious force acting on the body as a result of using a non-inertial frame of reference; examples are the centrifugal and Coriolis forces that appear in rotating coordinate systems. Also known as effective force", [2].

There is a negative sign on the right hand side of the Eq. (1.73) because the inertial force always acts in the opposite direction to the acceleration of the body.

1.7.2.2. Mass Moment of Inertia and Inertia Law
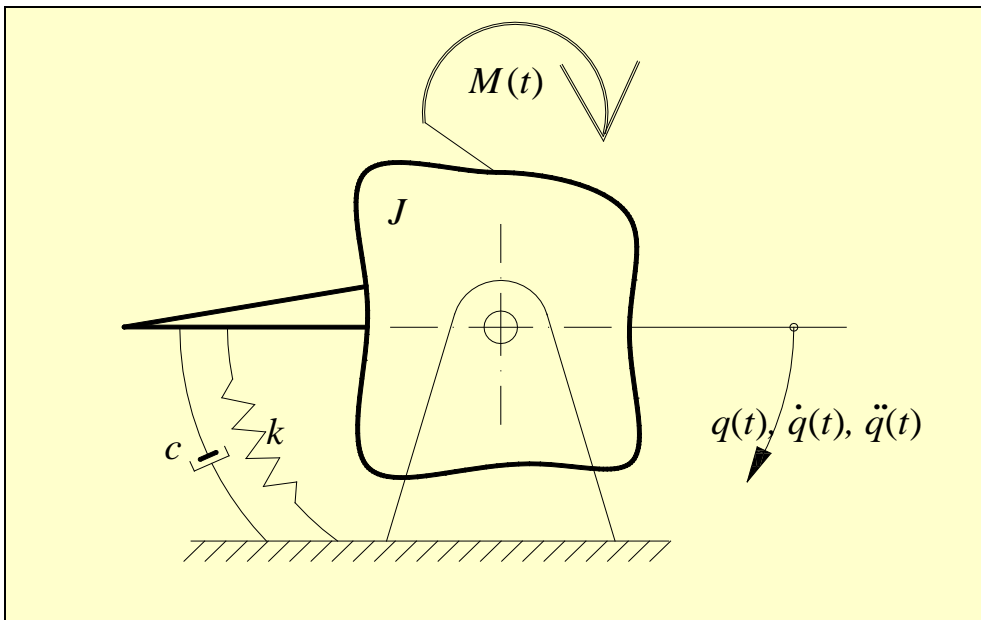
Fig. 1.10 Rotation of Rigid Mass.

In an inertial reference frame, the angular acceleration $\ddot{\alpha}$ of the mass moment of inertia J of a rigid body shown in Fig. 1.10 is, according to Newton's Second Law of Motion, proportional to the resultant moment of all the forces acting on the mass i.e.

$$\mathbf{M} = J\,\ddot{\alpha} \qquad (1.74)$$

In a non-inertial reference frame, the angular acceleration $\ddot{\alpha}$ of the mass moment of inertia J of a rigid body shown in Fig. 1.10 is proportional to the inertial moment

$$\mathbf{M}_m = -J\,\ddot{\alpha} \qquad (1.75)$$

There is a negative sign on the right hand side of the Eq. (1.75) because the inertial moment always acts in the opposite direction to the acceleration of the angular displacement.

Remark:

Mass moment of inertia J for the rigid body can be expressed as the product of constant density ρ of the body material and second moment of area I of the body i.e.

$$J = \rho I \qquad (1.76)$$

where

J	moment of mass inertia [kgm^2]
I	second moment of area [m^4] or second moment of volume [m^5]
$\rho = \dfrac{m}{V}$ or $\rho = \dfrac{m}{A}$	density of the body [kg/m^3] or density of the area [kg/m^2]
m	mass of the body [kg]
V	volume of the body [m^3]
A	area of body [m^2]

1.7.2.3. Kinetic Energy

Kinetic energy is the energy which a body possesses because it is in motion. For the translational movement of one mass, kinetic energy can be written in form

$$E_k = \frac{1}{2}\,m\,v^2 = \frac{1}{2}\,m\,\dot{q}^2 \qquad (1.77)$$

and for rotational movement of one rigid body – in form

$$E_k = \frac{1}{2}\,J\,\omega^2 = \frac{1}{2}\,J\,\dot{q}^2 \qquad (1.78)$$
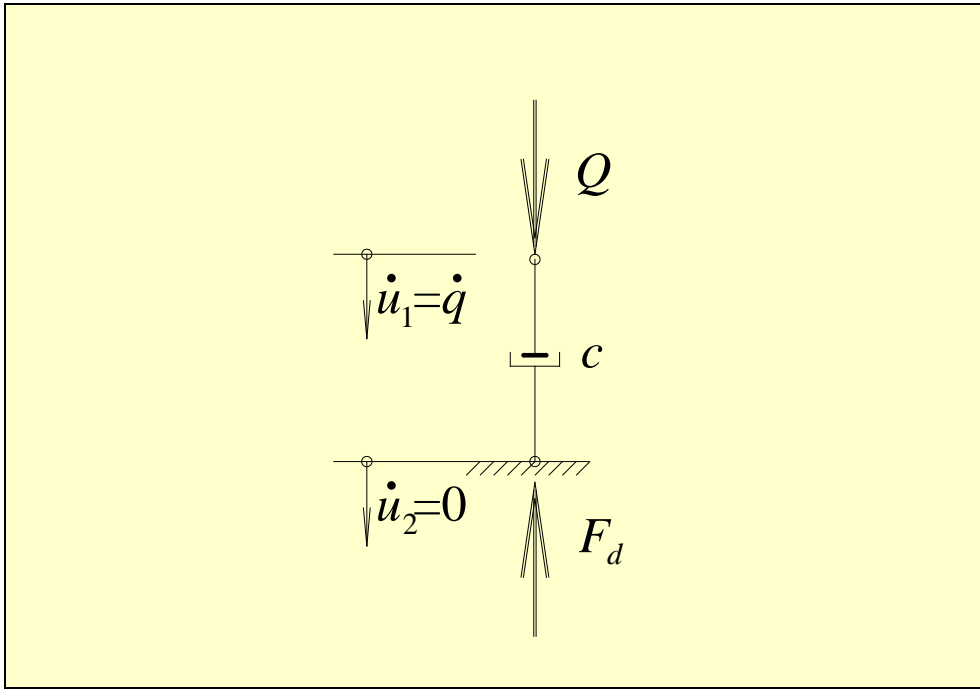
1.7.3. Damper

Fig. 1.11 Viscous damper.

1.7.3.1. Damping Law

In the viscous damper shown in Fig. 1.11 the applied force $\mathbf{Q}$ is proportional to the relative velocity of its connection points (in Fig. 1.11 this velocity is equal to velocity of displacement $\Delta\dot{\mathbf{u}} = \dot{\mathbf{q}}$). The resistance force (reaction) $\mathbf{F}_d = -\mathbf{Q}$ exerted by damping is described by the formula

$$\mathbf{F}_d = -c \cdot \Delta\dot{\mathbf{u}} = -c\,(\dot{\mathbf{u}}_1 - \dot{\mathbf{u}}_2) \tag{1.79}$$

where the minus sign means that the force vector is oppositely directed to the velocity of displacement vector $\Delta\mathbf{u} = \dot{\mathbf{u}}_1 - \dot{\mathbf{u}}_2$ (difference of velocity vectors of both ends of the spring along the spring axis). If velocity of displacement $\dot{\mathbf{u}}_1 = \dot{\mathbf{q}}$ and $\dot{\mathbf{u}}_2 = \mathbf{0}$, the Damping Law can be written in form

$$\mathbf{F}_d = -c\,\dot{\mathbf{q}} \tag{1.80}$$

The constant c is the damping coefficient, the characteristic parameter of the damper. The ideal damper is considered to have no mass, thus the force at one end is equal and opposite in direction to the force at the other end.

There are negative signs on the right hand sides of the Eqs. (1.79) and (1.80) since the resistance force always acts in the opposite direction to the displacement's velocity.

1.7.3.2. Rayleigh Dissipation Function

This function describes the power of resistance forces, i.e. the work of these forces in time, which occurs in systems performing small oscillations. These forces are assumed to be proportional to velocities. The Rayleigh dissipation function, also known as dissipation function, is given by

$$\Phi = \frac{1}{2}\, c\, \dot{q}^{\,2} \qquad (1.81)$$

1.7.4. Work of Acting Force

Work is the transfer of energy that occurs when a force acts on a body, and is calculated as a dot product of the force vector $\mathbf{F}$ and the displacement vector $\mathbf{q}$ of the point the force is acting on

$$W = \mathbf{F} \cdot \mathbf{q} = Fq\cos\theta \qquad (1.82)$$

where θ is the angle between the force and the displacement vectors.

If a body is moving in such a way that the force has a component in a direction perpendicular to the direction of the body's motion, the work of this component is equal to zero.

1.8. Springs and Dampers in Parallel or in Series

1.8.1. Springs

1.8.1.1. Springs in Parallel

Sometimes it is necessary to determine the equivalent (effective) spring constant k_e for a system in which two or more springs are arranged in parallel as shown in Fig. 1.12 or in series as in Fig. 1.13
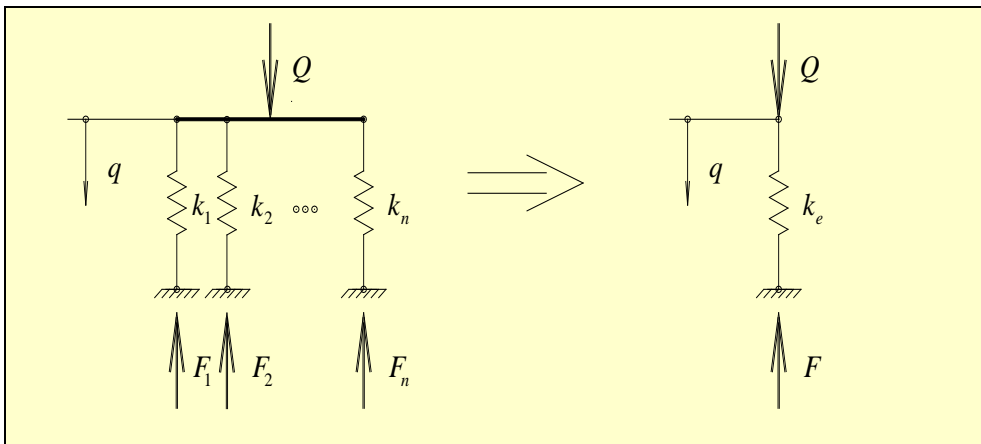
Fig. 1.12 Combination of springs – springs in parallel.

For springs in parallel, as shown in Fig. 1.14, the total reaction force is a sum of the forces' values in springs. According to Spring Law, Eq. (1.68), the expression can be written

$$F = F_1 + F_2 + \ldots + F_n = k_1 q + k_2 q + \ldots + k_n q = (k_1 + k_2 + \ldots + k_n)q = k_e q \qquad (1.83)$$

Finally the equivalent stiffness of the spring k_e is given by

$$k_e = k_1 + k_2 + \ldots + k_n \qquad (1.84)$$

which means that the stiffness of the equivalent spring is a sum of the stiffnesses of the springs assembled in parallel, i.e. In general, for n springs in parallel

$$k_e = \sum_{i=1}^{n} k_i \qquad (1.85)$$

Now, in accordance with Eq. (1.84) and (1.69) it can be written that

$$\frac{1}{\delta_e} = \frac{1}{\delta_1} + \frac{1}{\delta_2} + \ldots + \frac{1}{\delta_n} = \sum_{i=1}^{n} \frac{1}{\delta_e} \qquad (1.86)$$

1.8.1.2. Springs in Series
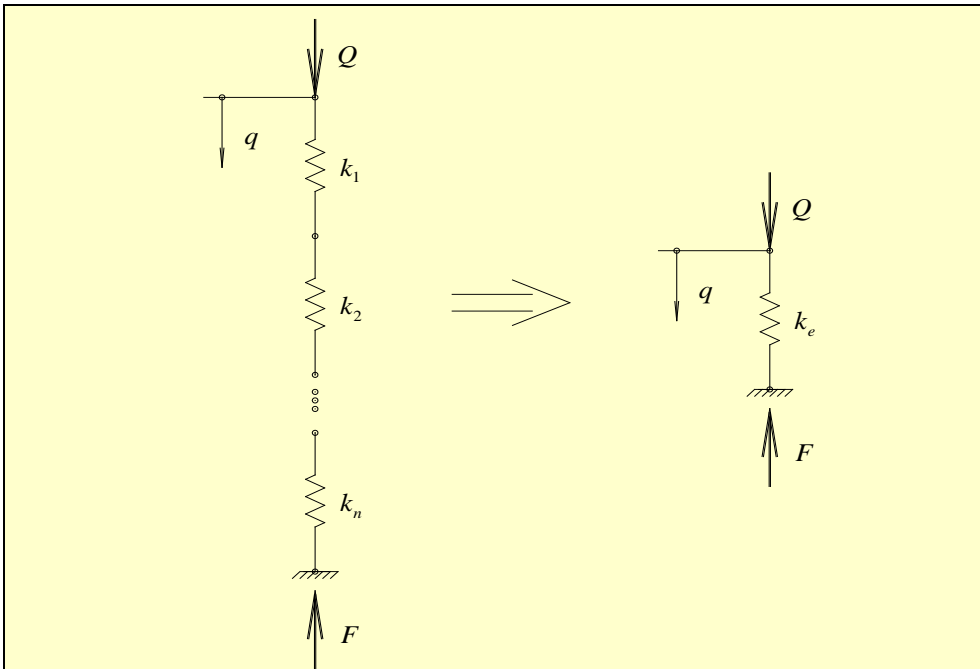
Fig. 1.13 Combination of springs – springs in series.

35

For springs assembled in series, as shown in Fig. 1.15, deflection (displacement) of the system is the sum of the deflection of the individual springs

$$q = \frac{F}{k_1} + \frac{F}{k_2} + \ldots + \frac{F}{k_n} = (\delta_1 + \delta_2 + \ldots + \delta_3)F = \delta_e F \qquad (1.87)$$

Finally the equivalent compliance of the spring δ_e is given by

$$\delta_e = \delta_1 + \delta_2 + \ldots + \delta_n = \sum_{i=1}^{n} \delta_i \qquad (1.88)$$

and using Eq. (1.69) one can achieve

$$\frac{1}{k_e} = \frac{1}{k_1} + \frac{1}{k_2} + \ldots + \frac{1}{k_n} \qquad (1.89)$$

and now it can be said that the inverse of the equivalent spring stiffness is a sum of the inverses of the springs' stiffnesses.

In general for n springs in series the equivalent spring constant may be obtained from

$$\frac{1}{k_e} = \sum_{i=1}^{n} \frac{1}{k_i} \qquad (1.90)$$

1.8.2. Dampers

The equivalent (effective) damper coefficient c_e for a system in which two or more dampers are arranged in parallel or in series can be appointed in a method analogical to the spring constant.

In general, for n dampers in parallel

$$c_e = \sum_{i=1}^{n} c_i \qquad (1.91)$$

and for n dampers in series, the equivalent damper coefficient c_e may be obtained from

$$\frac{1}{c_e} = \sum_{i=1}^{n} \frac{1}{c_i} \qquad (1.92)$$

1.9. Springs and Dampers neither in Parallel nor in Series

1.9.1. Equivalent Stiffness of One Spring

Let us consider one spring, whose ends can move as shown in Fig. 1.14,
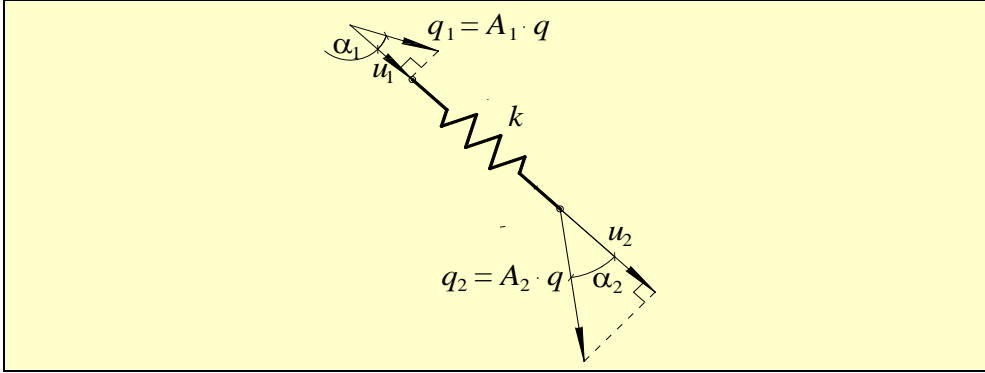
Fig. 1.14 A general case of a spring's deformation.

where q_1, q_2 are displacements of both ends of the spring, u_1, u_2 are their orthogonal projections onto the direction of the spring i.e. $u_1 = q_1 \cos \alpha_1$ and $u_2 = q_2 \cos \alpha_2$. Now, let us assume that displacements of ends of the spring q_1, q_2 depend on one generalized coordinate q according to the relations $q_1 = A_1 q$ and $q_2 = A_2 q$. Finally one can archive

$$u_1 = q\, A_1 \cos \alpha_1 \qquad \text{and} \qquad u_1 = q\, A_1 \cos \alpha_1 \tag{1.93}$$

The potential energy of the spring can be written in form (see Eq. (1.70))

$$E_p = \frac{1}{2} k (u_2 - u_1)^2 \tag{1.94}$$

Substituting the relationships Eq. (1.93) into Eq. (1.94) one receive

$$E_p = \frac{1}{2} k (A_2 \cos \alpha_2 - A_2 \cos \alpha)^2 q^2 = \frac{1}{2} k_e q^2 \tag{1.95}$$

Finally the formula for equivalent spring constant k_e may be formulated

$$k_e = k (A_2 \cos \alpha_2 - A_2 \cos \alpha)^2 \tag{1.96}$$

If displacement of one end of spring is equal to zero ($u_1 = 0$) and $A_2 = 1$ i.e. $q_2 = q$ the above simplified formula has the form

$$k_e = k \cos^2 \alpha_2 \tag{1.97}$$

1.9.2. Equivalent Damping Coefficient of One Damper

The same reasoning in the case of energy dissipation function leads to formulas for equivalent damping coefficient c_e of one damper

$$c_e = c\,(A_2 \cos\alpha_2 - A_2 \cos\alpha)^2 \qquad (1.98)$$

and if displacement velocity of one end of damper is equal to zero ($u_1 = 0$) and $A_2 = 1$ i.e. $\dot{q}_2 = \dot{q}$

$$c_e = c \cos^2\alpha_2 \qquad (1.99)$$

1.9.3. Springs or Dampers in Arbitrary Systems

Illustrative Example 1.4

Sometimes it is necessary to determine the equivalent spring constant k_e for a system in which two or more springs are arranged neither in parallel nor in series (for example Fig. 1.15 and Fig. 1.16).
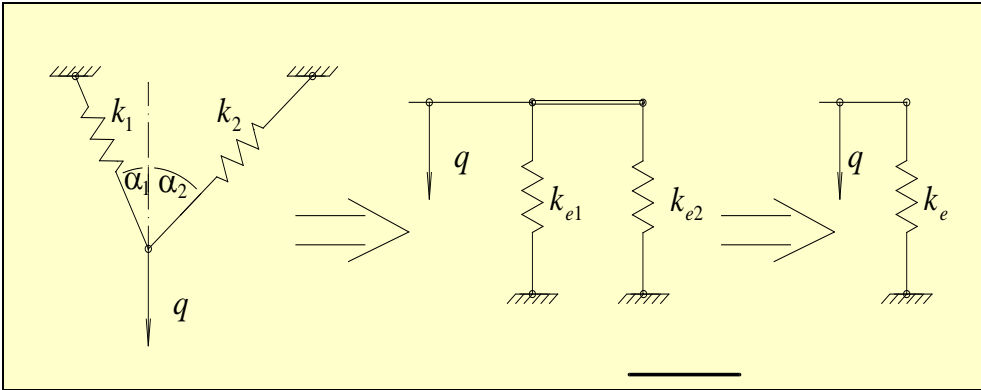
Fig. 1.15 Combination of springs – neither in parallel nor in series (quasi-parallel).

The potential energy of the system shown in Fig. 1.15 can be written in form

$$E_p = \frac{1}{2}\,k_1\,u_1^2 + \frac{1}{2}\,k_2\,u_2^2 = \frac{1}{2}\,k_1 \cos^2\alpha_1\,q^2 + \frac{1}{2}\,k_2 \cos^2\alpha_2\,q^2 =$$
$$= \frac{1}{2}\,(k_{1e} + k_{2e})\,q^2 = \frac{1}{2}\,k_e\,q^2 \qquad (1.100)$$

In general, for *n* springs the equivalent spring constant may be obtained from

$$k_e = \sum_{i=1}^{n} k_{ei} \qquad (1.101)$$

Illustrative Example 1.5

In situation shown in Fig. 1.16 the combination of springs corresponds to a set of equivalent springs in series.
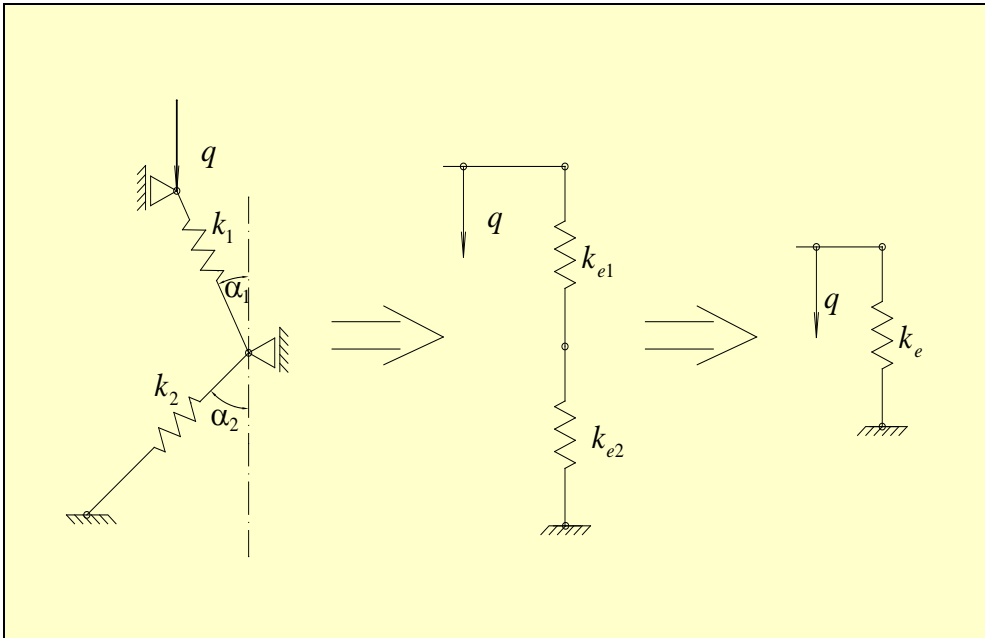

Fig. 1.16 Combination of springs – neither in parallel nor in series (quasi-serial)

Thus the equivalent stiffness of a system can be calculated from formula

$$\frac{1}{k_e} = \sum_{i=1}^{n} \frac{1}{k_{ei}}$$

(1.102)

where

$$k_{ei} = k_i \cos^2 \alpha_i$$

(1.103)

By analogy, the same procedure can be used to dampers connected in the same way as is shown in Fig. 1.15 and Fig 1.16. With the use of formulas

$$c_e = \sum_{i=1}^{n} c_{ei}$$

(1.104)

or

$$\frac{1}{c_e} = \sum_{i=1}^{n} \frac{1}{c_{ei}}$$

(1.105)

respectively, can be calculated the equivalent damper coefficient c_e.

1.10. Free Body Diagram (FBD)

Before proceeding to perform dynamic analysis for a dynamic system (Fig. 1.17) using d'Alembert's Principle (see Chapter 1.11), one should always create and draw a Free Body Diagram (Fig. 1.18).
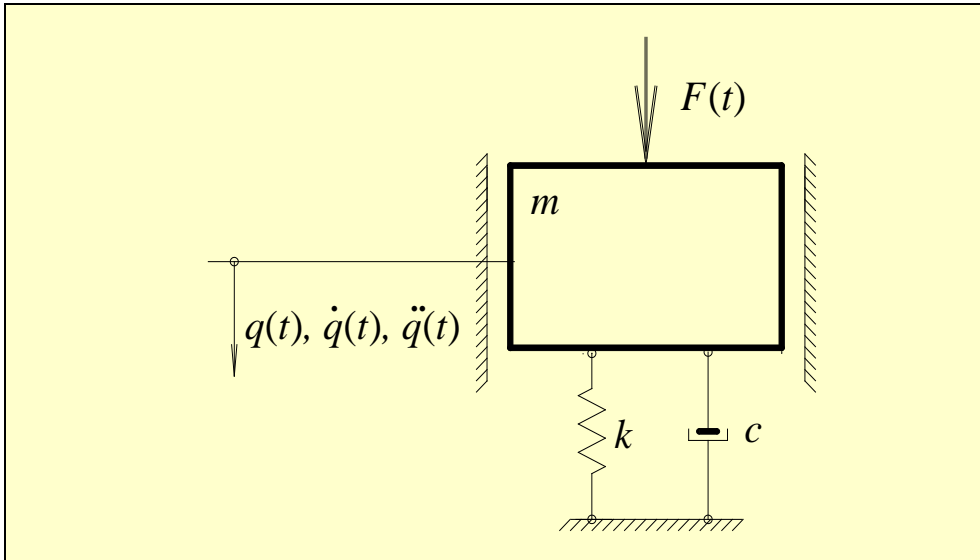
Fig. 1.17 Single-degree-of-freedom system with viscous damping, excited by force acting on mass.
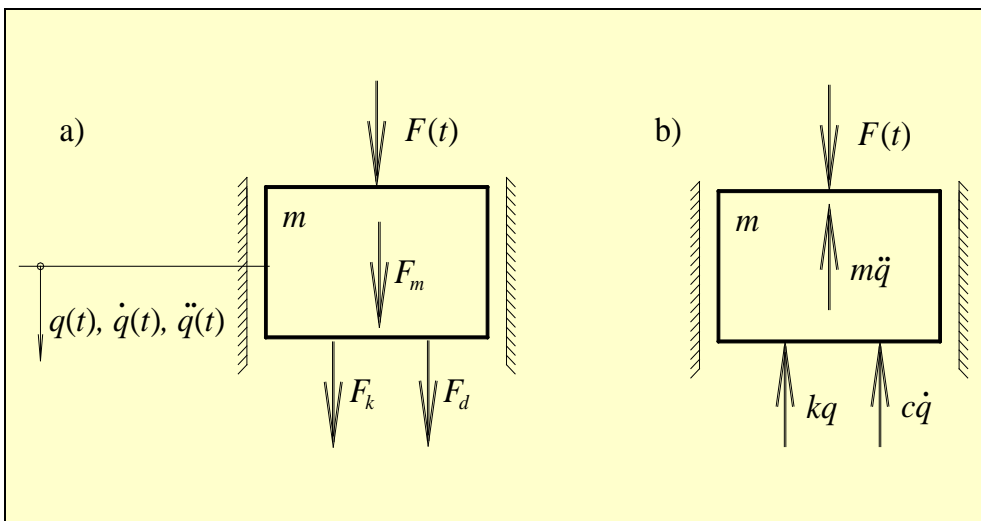

Fig. 1.18 Free body diagrams for a single-degree-of-freedom system with viscous damping, excited by force acting on mass.

In Fig. 1.18a can be seen a sketch of the body isolated from all other bodies, in which all the forces external to the body are shown. If d'Alembert's Principle is taken into account, the inertia forces also ought to be shown in this sketch. While creating an FBD, one should take into account and mark the characteristics of the body, such as mass, moment of inertia, etc. It is also necessary to mark the displacement vector. The direction and sense of this vector automatically determines the direction and sense of the axis of the frame of reference – so direction of this vector is by definition positive.

All other vectors:

$\dot{\mathbf{q}}$ – velocity vector

$\ddot{\mathbf{q}}$ – acceleration vector

$\mathbf{F}_m$ – inertia force vector

$\mathbf{F}_d$ – resistance force vector

$\mathbf{F}_k$ – restoring force vector

$\mathbf{F}(t)$ – applied force vector

should be marked with arrows, indicating their senses, which must be the same as the sense of the displacement $\mathbf{q}$ (generalized coordinate). In reality, the sense of these vectors may be opposite to the one shown in the drawing in Fig. 1.18a, as in Eqs. (1.68), (1.73) and (1.80). Instead of the forces, the right-hand sides of Eqs. (1.68), (1.73) and (1.80) (with a negative sign) should be introduced into the equation of dynamic equilibrium.

There is also another method of creating an FBD. The senses of negative vectors ought to be changed in the diagram to their opposites, which means that the arrows indicating the sense of vector introduced into the diagram are to be drawn with their real sense, see Fig. 1.18b. The values of the forces should be introduced into the equation of dynamic equilibrium with positive signs.

The weight (gravity force) of the body $\mathbf{G} = m\mathbf{g}$ can also be shown in this diagram if necessary. However, then it is also necessary to mark the static displacement $\mathbf{q}_{st}$ and the static restoring force $\mathbf{F}_r$.

1.11. D'Alembert's Principle

An alternative approach to Newton's Second Law of Motion is to make use of d'Alembert's Principle to obtain the equation of motion. This principle states that when the material points move in accordance to the constraints, the difference of applied forces and inertial forces in a dynamic system does no virtual work.

$$\delta W = \sum_i (\mathbf{F}_i + \mathbf{F}_{m,i}) \cdot \delta \mathbf{r}_i = 0 \tag{1.106}$$

where

$\mathbf{F}_i$ is the vector of applied force acting on point i of the system,

$\mathbf{F}_{m,i} = -m_i \ddot{\mathbf{q}}_i$ is the vector of inertial force of point i of the system,

m_i is the mass of the particle i in the system

$\ddot{\mathbf{q}}_i$ is the acceleration of the particle i in the system

$\delta \mathbf{r}_i$ is the virtual displacement of point i of the system, consistent with the constraints

The dot between vectors in formula Eq. (1.106) means a dot product of the vectors. The vector of the inertial force is the product of the mass and its acceleration. The negative sign means that the sense of the inertia force vector is always opposite to the sense of the vector of mass acceleration.

In a static case, when $\mathbf{F}_{m,i} = -m_i \ddot{\mathbf{q}}_i = 0$, d'Alembert's Principle Eq. (1.106) takes the form of Principle of Virtual Work.

D'Alembert also showed that one can transform an accelerating rigid body into an equivalent static system by adding the inertial force.

D'Alembert's Principle

"a system may be set in a state of dynamic equilibrium by adding to the external forces a fictitious force that is commonly known as the inertial force", [4].

The application of d'Alembert's Principle makes it possible to use equations of dynamic equilibrium in obtaining the equation of motion.

The most important feature of d'Alembert's Principle is its simplicity – in the formulation of principle the arbitrary virtual displacements are assumed to be in directions that are orthogonal to the constraint forces, so the constraint forces do no work and unknown reactions are not present in equations. It is not necessary to calculate these forces to find equations of motion of the system

Illustrative Example 1.6
Equation of Motion
According to d'Alembert's Principle it is possible to formulate the equation of motion of a single-degree-of-freedom viscously damped system excited in forced vibration by a force acting on the mass, Fig. 1.17. Especially in simple systems, the free body diagram is very useful in achieving the equation of motion, Fig. 1.18.

D'Alembert's Principle states that a dynamic system may be set in a state of dynamic equilibrium by adding to the external forces an inertial force (IF). Looking at Fig. 1.18a, the summation of forces acting on the mass in the direction of the generalized coordinate q gives directly

$$\sum \mathbf{F}_q = 0 \qquad (1.107)$$

Thus, we achieve

$$\mathbf{F}_m + \mathbf{F}_d + \mathbf{F}_k + \mathbf{F}(t) = \mathbf{0} \qquad (1.108)$$

Now, using the expressions described by Eqs. (1.68), (1.73), (1.80) it is possible to write down

$$-m\ddot{\mathbf{q}} - c\,\dot{\mathbf{q}} - k\,\mathbf{q} = -\mathbf{F}(t) \qquad (1.109)$$

Multiplying the above equation by (–1) one can achieve

$$m\ddot{\mathbf{q}} + c\,\dot{\mathbf{q}} + k\,\mathbf{q} = \mathbf{F}(t)$$ (1.110)

This equation is the vector differential equation of motion for an SDOF system with viscous damping and force applied to mass.

The scalar form of Eq. (1.110) is

$$m\ddot{q} + c\,\dot{q} + k\,q = F(t)$$ (1.111)

The same equation one can achieved summing the force in BD shown in Fig. 1.18b.

In this case, the application of d'Alembert's Principle may seem trivial. In more complex systems, however, the problem becomes more difficult, and while the use of d'Alembert's Principle is still possible, it becomes too complicated to be practical. This is due to the fact that in this instance, d'Alambert's Principle would have to be applied together with the Principle of Virtual Work. Because of this, the Lagrangian equation will be more convenient to the solution of such complex systems.

Single-Degree-of-Freedom (SDOF) Systems

2. Single-Degree-of-Freedom (SDOF) Systems

2.1. Degrees of Freedom (DOF)

Degrees of freedom are the ways in which the space configuration of a mechanical system may change, i.e. the independent movements the system can possibly undergo. If only one of these independent movements is possible, the system is called a Single-Degree-Of-Freedom (SDOF) System

2.2. Lagrangian Equation

As has been stated in Chapter 1.11, formulating a differential equation of motion for a vibrating system by applying d'Alambert's Principle is sometimes complicated, as it requires the determination of all the forces acting on the masses in an SDOF system. Therefore, in such more complex cases, it is often easier to derive this equation of motion in terms of the energies of the system with the use of the Lagrangian equation, [5]

$$\frac{d}{dt}\frac{\partial E_k}{\partial \dot{q}} - \frac{\partial E_k}{\partial q} + \frac{\partial \Phi}{\partial \dot{q}} + \frac{\partial E_p}{\partial q} = \frac{\partial W}{\partial q} \qquad (2.1)$$

where

E_k total kinetic energy of the system

E_p total potential energy of the system

Φ Rayleigh Dissipation Function

W virtual work of external force

q , $\dot{q}$ generalized coordinate – displacement and velocity

In the case of small vibration around the equilibrium point, the kinetic energy in general does not depend on generalized displacements q. Then

$$\partial E_k / \partial q = 0 \qquad (2.2)$$

and Lagrangian equation has then the form used more often in typical situations

$$\frac{d}{dt}\frac{\partial E_k}{\partial \dot{q}} + \frac{\partial}{\partial \dot{q}} + \frac{\partial E_p}{\partial q} = \frac{\partial W}{\partial q} \qquad (2.3)$$

2.3. Equation of Motion

After the substitution of formulas for: potential energy Eq. (1.71), kinetic energy Eq. (1.77), Rayleigh dissipation function Eq. (1.81) and work of acting force Eq. (1.82) into Eq. (2.4), the result can be written in a form

$$m\ddot{q} + c\,\dot{q} + k\,q = F(t) \qquad (2.4)$$

identical with the Eq. (1.111), which was achieved by using d'Alembert's Principle.

2.4. Undamped Systems

In ideal conditions, it is theoretically possible that a simple oscillator will never stop vibrating with a constant amplitude at its natural frequency. Such conditions are created when there is no damping at all. However, in practice it is impossible to eliminate all damping – therefore, the assumption that such conditions exist can only be warranted when the damping is negligibly small. Even so, the idealized solutions for such a case are of interest from the theoretical and didactical point of view.

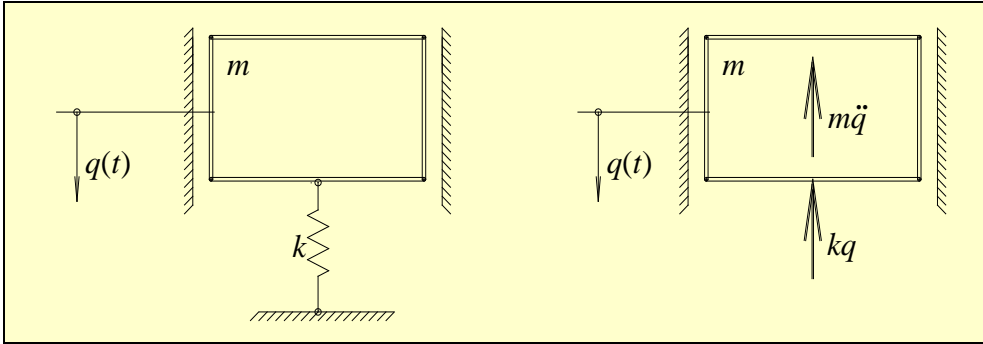The simplest possible vibratory system is shown in Fig. 2.1.

Fig. 2.1 Undamped single-degree-of-freedom system

This system consists of a mass m attached to the foundation by means of a spring with stiffness k. The mass can only move in one direction (translational motion) – the direction of the q axis (generalized coordinate). For this reason such a system is called a single-degree-of-freedom system. Free vibration of such systems are discussed below.

2.4.1. Homogeneous Equation of Motion and its Solution

In accordance with Eq. (2.4) the equation of motion for undamped SDOF systems can be written in form

$$m\ddot{q} + k\,q = 0 \tag{2.5}$$

where $q = 0$ defines the equilibrium position of the mass. This is the second order differential equation with constant coefficients.

The solution of the above equation can be written in the analytical form. The first step to achieve the analytical solution is to put in the Eq. (2.5)

$$q = e^{\lambda t} \tag{2.6}$$

The characteristic equation obtained from Eq. (2.5) is

$$m\lambda^2 + k = 0 \tag{2.7}$$

There are two roots of this equation $\lambda_1 = i\omega_n$ and $\lambda_2 = -i\omega_n$, where $i = \sqrt{-1}$ and

$$\omega_n = \sqrt{k/m}$$ (2. 8)

is known as the natural frequency of the system.

A total integral (general solution) of Eq. (2.5) has the form

$$q(t) = q_S \sin \omega_n t + q_C \cos \omega_n t$$ (2.9)

where q_S, q_C are the constants of integration.

2.4.2. Free Vibration

Free vibration occurs when a state of static equilibrium is disturbed, i.e. initial non-zero conditions are introduced into the system and there are no external excitation forces acting on the system, Fig. 2.1. This is a physical phenomenon described by the solution of Eq. (2.9) with two initial conditions, $q(0) = q_o$, $\dot{q}(0) = \dot{q}_o$ which determine constants of integration, i.e.

$$\begin{aligned} q(0) = q_C = q_o & \quad \rightarrow \quad q_C = q_o \\ \dot{q}(0) = \omega_n q_S = \dot{q}_o & \quad \rightarrow \quad q_S = \dot{q}_o/\omega_n \end{aligned}$$ (2.10)

Finally the solution (displacement and velocity) has the form

$$q(t) = q_o \cos \omega_n t + \dot{q}_o \sin \omega_n t/\omega_n$$ (2.11)

$$\dot{q}(t) = -q_o \omega_n \sin \omega_n t + \dot{q}_o \cos \omega_n t$$ (2.12)
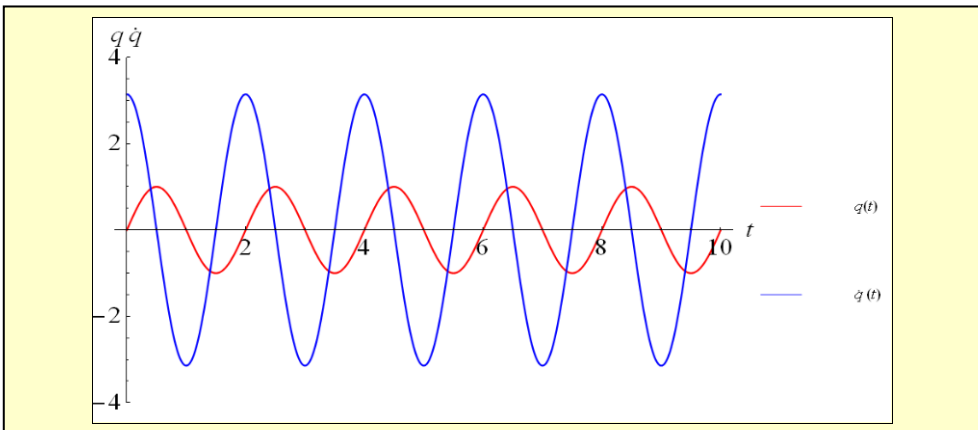
The exemplary free vibration can be seen in Fig. 2.2

Fig. 2.2 Free vibration – undamped single-degree-of-freedom system:
$\omega = 3[rad/s]$, $q_o = 1[m]$, $\dot{q}_o = 2[m/s]$

2.4.3. Forced Vibration

The vibration of a system is forced if the response is dependent on the excitation. The non-resonant vibration is always steady-state when the excitation is periodic and continuous. The focus of the present chapter is harmonic excitation, which is important for two reasons. On the one hand, a large number of civil engineering structures is subjected to harmonic excitation produced by rotating machinery. On the other, the mathematical description of such excitation is relatively easy, as the solution of the equation of motion in this case is obtained in an analytical form. Even if excitation is not harmonic, it can be decomposed into a Fourier series (a series of harmonic functions). Through the superposition of individual responses of the system to these harmonic components of external excitation, we achieve a response of the system to the non-harmonic excitation.

2.4.3.1. Response to Harmonic Load

When the sinusoidal force $F(t) = F_o \sin \omega t$ is applied to the mass in the undamped single-degree-of-freedom system, Fig. 2.3, the differential equation of motion is
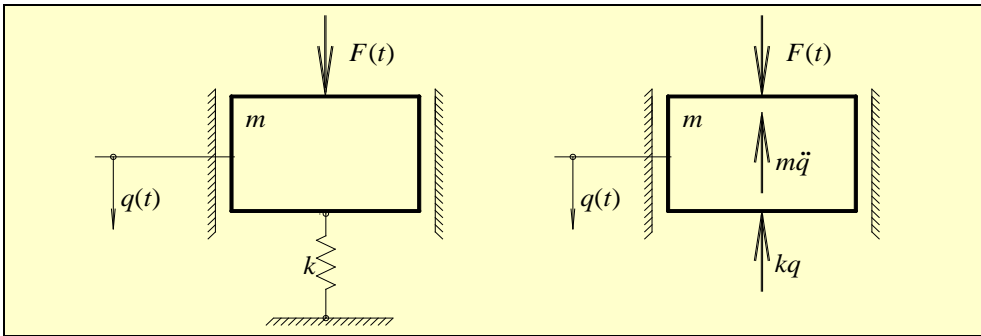

Fig. 2.3 Undamped single-degree-of-freedom system excited in forced vibration by a harmonic force acting on mass

$$m\ddot{q} + k\,q = F_o \sin \omega t \tag{2.13}$$

The solution of this equation is

$$q(t) = q_S \sin \omega_n t + q_C \cos \omega_n t + \frac{F_o}{k}\frac{1}{1 - \omega^2/\omega_n^2}\sin \omega t \tag{2.14}$$

This total (general) solution to the linear differential equation Eq. (2.13) is the sum of the general solution (also called a complementary solution – the sum of the first two components on the right-hand side of Eq. (2.14)) of the related (reduced) homogeneous equation Eq. (2.5), and the particular solution (integral – the third component on the right-hand side of Eq. (2.14)). The complementary solution depends on the initial conditions. The particular solution depends on the forces of excitation only (does not depend on initial conditions).

The total (general) solution is related to vibrations with the undamped natural frequency ω_n. The coefficients q_C, q_C may be found from a comparison of $q(t)$, Eq. (2.14), and its first derivatives $\dot{q}(t)$ at the time-point $t = 0$, with the initial conditions $q(0) = q_o$, $\dot{q}(0) = \dot{q}_o$, i.e. from the set of equations

$$\boxed{q(0) = q_C = q_o} \qquad \boxed{\dot{q}(0) = q_S \omega_n + \frac{F_o}{k} \frac{\omega}{1 - \omega^2/\omega_n^2}} \qquad (2.15)$$

The exemplary solutions if the mass is initially at rest in the equilibrium position of the system (i.e., $q(0) = 0$ and $\dot{q}(0) = 0$) at time $t > 0$ is

$$\boxed{q(t) = \frac{F_o}{k} \frac{1}{1 - \omega^2/\omega_n^2} \left(\sin \omega t - \frac{\omega}{\omega_n} \sin \omega_n t \right)} \qquad (2.16)$$
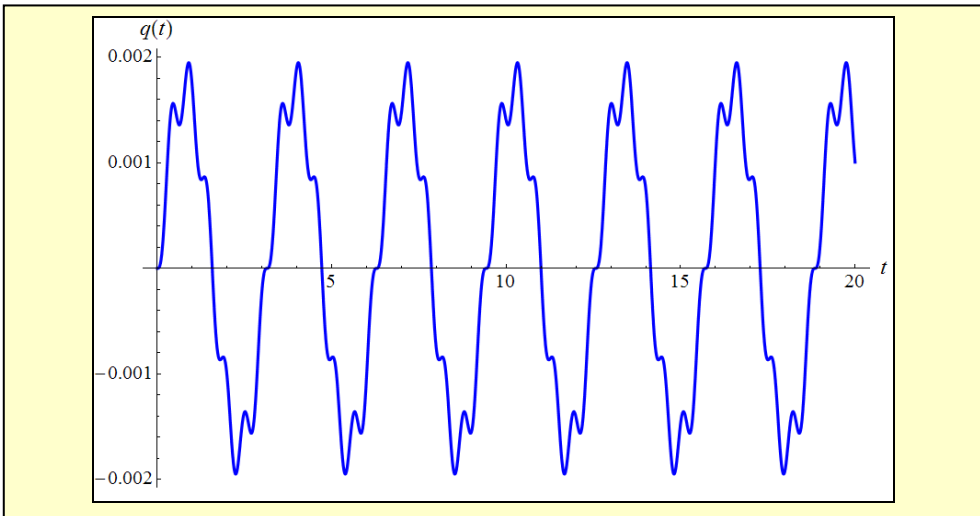
This solution can be seen in Fig. 2.4

Fig. 2.4 Exemplary solutions for an undamped single-degree-of-freedom system excited in forced vibration by harmonic force acting on mass,
$F_o = 10\,\text{kN}, \quad k = 10^6\,N/m, \quad \omega = 12\,\text{rad/s}, \quad \omega_n = 2\,\text{rad/s}$

In reality, the situation is different because of damping, which is always present in a physical system. Due to its effects, the vibration at natural frequency ω_n fades gradually. With its fading, the condition of equilibrium is fulfilled, and only a steady-state vibration at forcing frequency ω remains:

$$\boxed{q(t) = \frac{F_o}{k} \frac{1}{1 - \omega^2/\omega_n^2} \sin \omega t} \qquad (2.17)$$

This vibration exists as long as the force $F(t)$ is applied to the system.

2.4.3.2. Force Transmission to Foundation

Transmissibility v_T is defined as the measure of the ability of a system either to amplify or to suppress an input vibration, equal to the ratio of the response amplitude of the system in steady-state forced vibration to the excitation amplitude; the ratio may be in forces, displacements, velocities, or accelerations. [1, 2]

In the analyzed case (when the spring is the only element connecting the mass to the foundation), the force transmitted to the foundation is directly proportional to the spring deflection $F_T = F_k = kq$.

Substituting $q(t)$ from Eq. (2.17) one can achieve

$$v_T = \frac{F_T}{F} = \frac{am\,F_T}{am\,F} = \frac{1}{1 - \omega^2 / \omega_n^2} = \frac{1}{1 - \eta^2} \qquad\qquad \eta = \frac{\omega}{\omega_n} \qquad (2.18)$$

En exemplary response factor curve (transmissibility v_T) of an undamped single-degree-of-freedom system excited in forced vibration by a harmonic force acting on mass can be seen in Fig. 2.5a.
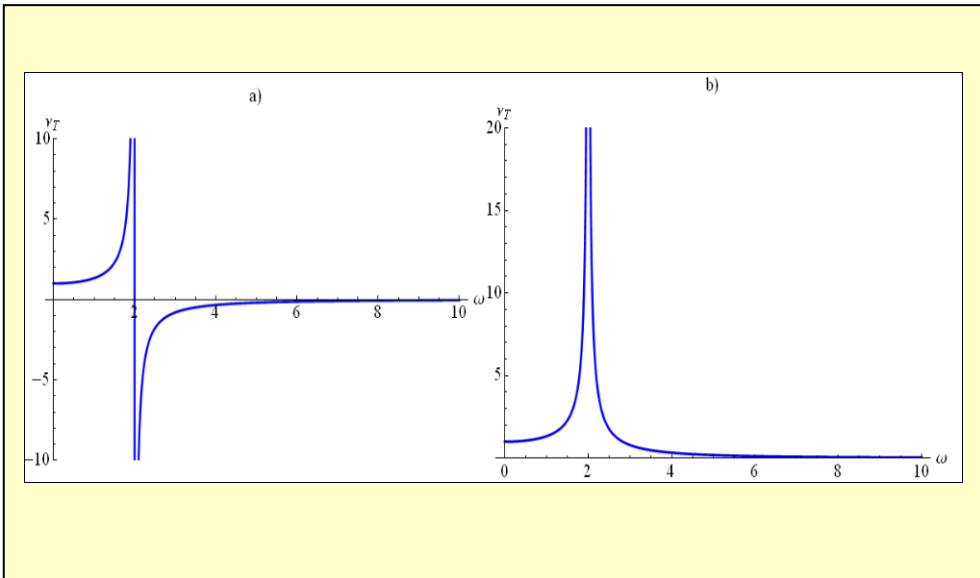

Fig. 2.5 Undamped single-degree-of-freedom system excited in forced vibration by a harmonic force acting on mass $\omega_n = 2\,\mathrm{rad/s}$: a) transmissibility; b) absolute value of transmissibility

Usually, the more convenient way of presenting the response factor is a curve of absolute value of transmissibility $|v_T|$. The response factor curve for $|v_T|$ is shown in Fig. 2.5b.

2.4.3.3. Resonant Vibrations

When a resonance occurs ($\omega = \omega_n$), the expression Eq. (2.16) describing steady-state vibration becomes indeterminate. In such a situation, a mathematical analysis of the problem leads to a different solution. A detailed solution to the problem may be found in [6]. The accurate solution $q(t)$ in this case has a mathematical form in which the variable t is the multiplier of the harmonic component:

$$q(t) = \frac{F_o \omega}{2k} t \cos \omega t - \frac{F_o}{2k} \sin \omega t \qquad (2.19)$$

According to the above solution, the amplitude of $q(t)$ increases linearly in time, reaching an infinitely great value only after an infinitely great time (see Fig. 2.6).
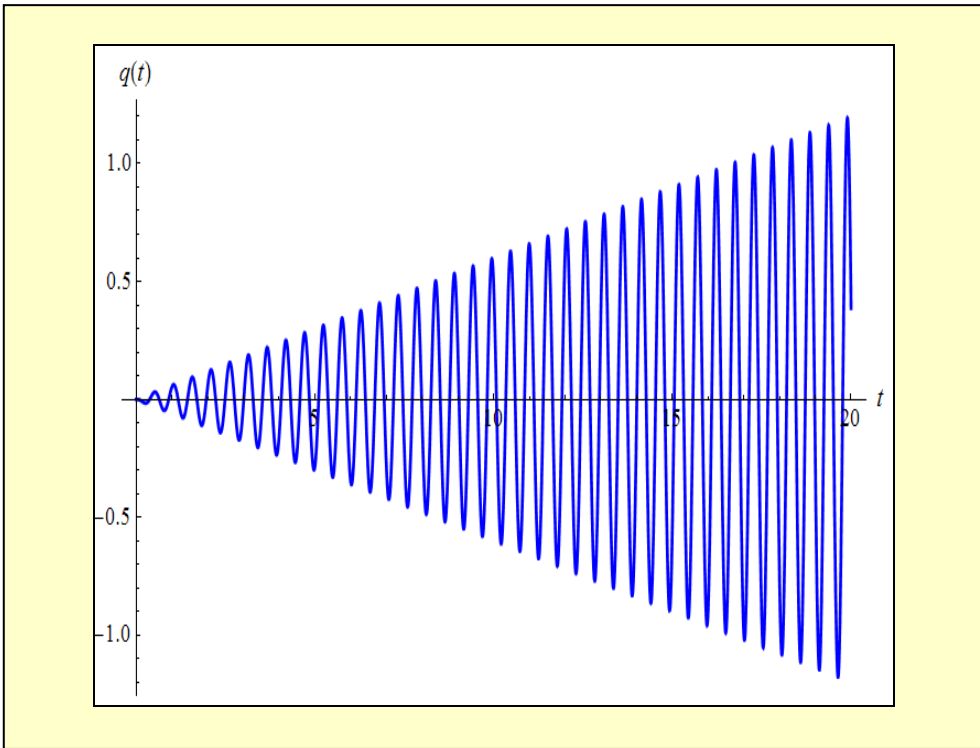

Fig. 2.6 Resonant vibration of an undamped single-degree-of-freedom system excited in forced vibration by a harmonic force acting on mass,
$$F_o = 10 \, \text{kN}, \quad k = 10^6 \, N/m, \quad \omega = \omega_n = 12 \, \text{rad/s}$$

2.4.3.4. Response to Harmonic Motion of Foundation

Let us consider the case of forced vibration excited by a continuing harmonic motion of the foundation (Fig. 2.7).
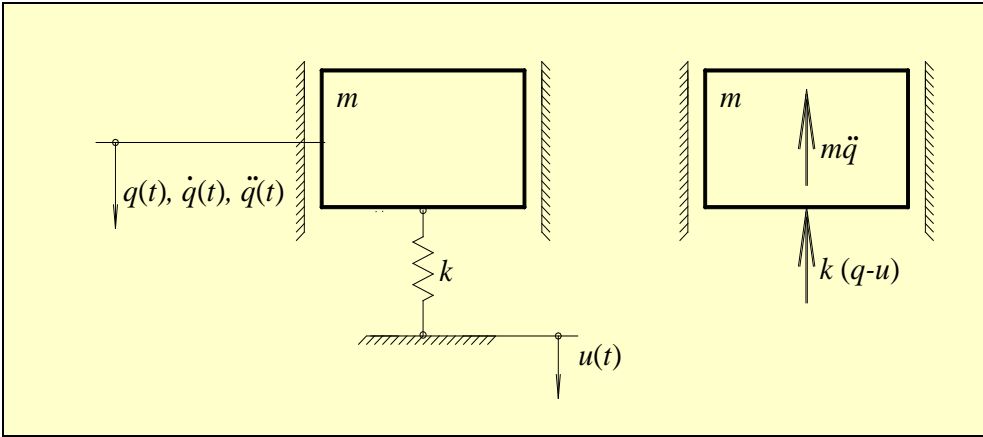
Fig. 2.7 Undamped single-degree-of-freedom system excited in forced vibration by motion of foundation

The differential equation of motion for the system, Fig. 2.7, excited by a continuing motion $u(t) = u_o \sin \omega t$ of the foundation, with accordance to Eqs. (1.67) and (2.5), is

$$m\ddot{q} + k(q - u_o \sin \omega t) = 0 \qquad (2.20)$$

The solution of this equation is

$$q(t) = q_S \sin \omega_n t + q_C \cos \omega_n t + \frac{u_o}{1 - \omega^2 / \omega_n^2} \sin \omega t \qquad (2.21)$$

where $\omega_n = \sqrt{k/m}$ and the coefficients q_S and q_C are determined by the velocity and displacement of the mass, respectively, at time $t = 0$. Although it does not result from the formula in Eq. (2.21), when damping occurs the general integral eventually fades entirely. As has been stated in the previous chapter, only the particular integral, i.e. the steady-state vibration, remains. Hence, the ratio of amplitudes is defined in the following terms:

$$\frac{\operatorname{am} q}{\operatorname{am} u} = v_T = \frac{1}{1 - \omega^2 / \omega_n^2} \qquad (2.22)$$

From a comparison between Eq. (2.22) and Eq. (2.18) it follows that in forced vibration of an undamped SDOF system, the force transmissibility, and the motion transmissibility are the same.

An exemplary response factor curve (transmissibility v_T) of an undamped single-degree-of-freedom system excited in forced vibration by motion of foundation can be seen in Fig. 2.5a, and the response factor curve for absolute value of transmissibility $|v_T|$ – in Fig. 2.5b.

2.5. Damped Systems

As has already been noted, in reality every physical system undergoing motion contains frictional forces, otherwise referred to as damping forces. These forces are responsible for the loss of energy from the system – the energy is mostly transformed into heat. As the mechanism of this transformation has not yet been fully accounted for, an analysis of dynamic systems requires a number of assumptions to be made on the basis of practical experience.

2.5.1. Viscously Damped System

The simplest way to take into account the damping forces in dynamic analysis is to assume that these forces are proportional to the magnitude of the velocity, and opposite in direction to the direction of motion. This type of damping is known as viscous damping. The assumption of viscous damping is not always realistic, but it is made often nonetheless for two reasons. Firstly, in civil engineering structures damping is usually very low, and so the kind of damping assumed is of no great importance. Secondly, the mathematical analysis of a system with this kind of damping is relatively simple.

In Fig. 2.9 one can see a single degree-of-freedom system with a mass m, a spring with stiffness k, and a viscous damper with damping coefficient c.
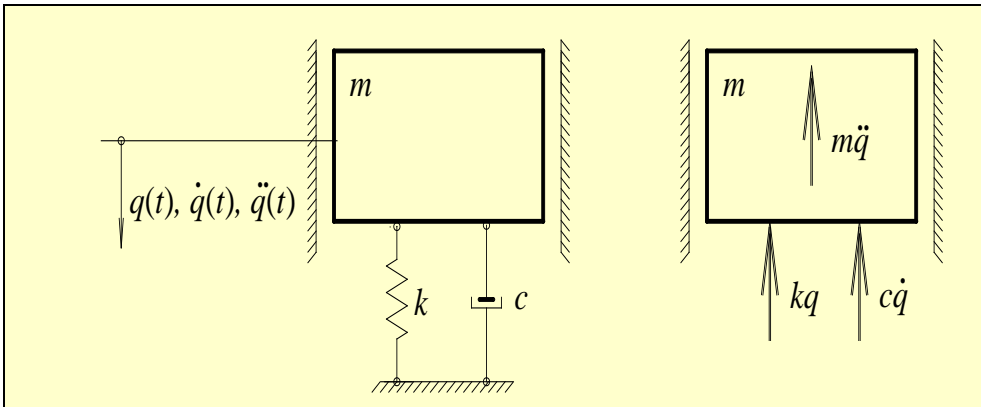

Fig. 2.8 Single-degree-of-freedom system with a viscous damper

2.5.2. Homogeneous Equation of Motion and its Solution

The differential equation of motion of mass m , corresponding to Eq. (1.111) is

$$m\ddot{q} + c\,\dot{q} + k\,q = 0 \qquad (2.23)$$

The solution of the above equation can also be written in the analytical form. The first step to achieve the analytical solution is to put into the Eq. (2.23)

$$q = e^{\lambda t} \qquad (2.24)$$

The characteristic equation obtained from Eq. (2.23) is

$$m\lambda^2 + c\lambda + k = 0 \qquad (2.25)$$

There are two roots of this equation $\lambda = -\alpha\omega_n + i\omega_d$ and $\lambda = -\alpha\omega_n - i\omega_d$, where

$$\omega_d = \omega_n\sqrt{1-\alpha^2} < \omega_n \qquad (2.26)$$

is the natural angular frequency of damped vibration – damping frequency.

The period of damped vibration is given by

$$T_d = \frac{2\pi}{\omega_d} = \frac{2\pi}{\omega_n\sqrt{1-\alpha^2}} = \frac{T_n}{\sqrt{1-\alpha^2}} > T_n \qquad (2.27)$$

and the damping ratio (fraction of critical damping) α is defined by

$$\alpha = \frac{c}{c_{cr}} = \frac{c}{2\sqrt{km}} \qquad (2.28)$$

where the critical damping coefficient $c_{cr} = 2\sqrt{km} = 2m\omega_n$.

Now, Eq. (2.23) can be written in the commonly used form

$$\ddot{q} + 2\alpha\omega_n\,\dot{q} + \omega_n^2\,q = 0 \qquad (2.29)$$

and its general integral (solution) has the form

$$q(t) = e^{-\alpha\omega_n t}(q_S\sin\omega_d t + q_C\cos\omega_d t) \qquad (2.30)$$

where q_S, q_C are the constants of integration.

2.5.3. Free Vibration with Viscous Damping

Free vibration occurs when a state of static equilibrium is disturbed, i.e. initial non-zero conditions are introduced into the system and there are no external excitation forces acting on the system, Fig. 2.8. This is a physical phenomenon described by the solution Eq. (2.30) with two initial conditions, $q(0) = q_o$, $\dot{q}(0) = \dot{q}_o$ which determine constants of integration, i.e.

$$\begin{aligned} q(0) = q_C = q_o &\qquad \rightarrow \quad q_C = q_o \\ \dot{q}(0) = \omega_d\,q_S - \alpha\omega_n q_o = \dot{q}_o &\qquad \rightarrow \quad q_S = \frac{1}{\omega_d}\dot{q}_o + \frac{\alpha}{\sqrt{1-\alpha^2}}q_o \end{aligned} \qquad (2.31)$$

Finally the solution has the form

$$q(t) = e^{-\alpha\omega_n t}\left(q_o \cos\omega_d t + \frac{\dot{q}_o + q_o \alpha\omega_n}{\omega_d}\sin\omega_d t \right) =$$
$$= e^{-\alpha\omega_n t}\frac{\cos(\omega_d t - \beta)}{\cos\beta}q_o + e^{-\alpha\omega_n t}\frac{\sin\omega_d t}{\omega_n \cos\beta}\dot{q}_o \qquad (2.32)$$

where $\sin\beta = \alpha$ and $\cos\beta = \sqrt{1-\alpha^2}$. Exemplary solution can be seen in Fig. 2.9.
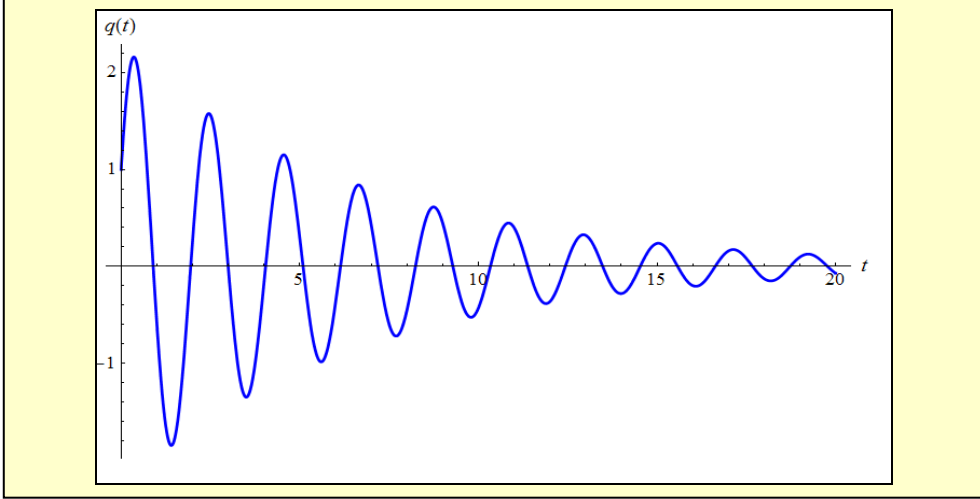

Fig. 2.9 Free vibration response for a damped system.

2.5.3.1. Underdamped System ($\alpha < 1$) – Less-Than-Critical Damping

In a situation when the damping of the system is less than critical $\alpha < 1$, a general solution of Eq. (2.23) has the form of Eq. (2.30).

2.5.3.2. Critically Damped System ($\alpha = 1$) – Critical Damping

In a situation when $c = c_{cr}$ there is no vibration and the solution of Eq. (2.23) is

$$q(t) = e^{-\alpha\omega_n t}(q_s + q_c\, t) \qquad (2.33)$$

2.5.3.3. Overdamped System ($\alpha > 1$) – Greater-Than-Critical Damping

In a situation when the damping of the system is greater than critical $\alpha > 1$, a total integral of Eq. (2.23) has the form

$$q(t) = e^{-\alpha\omega_n t}(q_S e^{t\omega_n\sqrt{\alpha^2-1}} + q_C e^{-t\omega_n\sqrt{\alpha^2-1}}) \qquad (2.34)$$

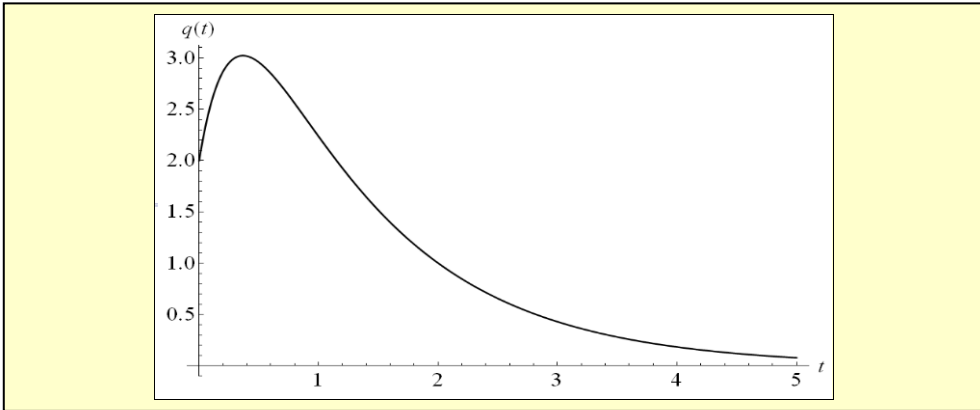In such a situation, the motion is non-oscillatory, Fig. 2.10.

Fig. 2.10 Free vibration response with critical damping.

This means that an overdamped system disturbed from its equilibrium position by introducing initial conditions does not perform vibration (at most, it will pass through the equilibrium position only once) and returns to the equilibrium position in a non-oscillatory way. The overdamped system's response is comparable to the response of a critically damped system, but the process of returning to the equilibrium state takes longer. The time required is proportional to the magnitude of the damping.

2.5.3.4. Logarithmic Decrement

A different way to determine the damping in a system is to specify the so called logarithmic decrement of damping. When $\alpha < 1$, the degree of damping may be defined in terms of two peak values of a free oscillation curve, Fig. 2.9, which can be derived either theoretically Eq. (2.31) or experimentally (record of the oscillatory motion). Let us consider any two maxima (i.e. value of $q(t)$ when $dq/dt = 0$) separated by n cycles of oscillation i.e. $q(t + nT_d) = e^{-\alpha\omega_n T_d} q(t)$. Then the natural logarithm of the ratio of these maxima is

$$\vartheta = \frac{1}{n} \ln \frac{q(t)}{q(t + nT_d)} = \alpha\omega_n T_d = \frac{2\pi\alpha}{\sqrt{1-\alpha^2}} \qquad (2.35)$$

where n is an integer number greater than or equal to 1 ($n \geq 1$).

From Eq. (2.35) one can find that

$$\alpha = \frac{\vartheta}{\sqrt{4\pi^2 + \vartheta^2}} \qquad (2.36)$$

For small values of α (less than about 0.1) the logarithmic decrement $\vartheta \ll 2\pi$, and an approximate relation between the damping ratio (fraction of critical damping) and the logarithmic decrement is

$$\vartheta = 2\pi\alpha \qquad (2.37)$$

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2.5.4. Forced Vibration with Viscous Damping

2.5.4.1. Response to Harmonic Load

Fig. 2.11 shows the case of a harmonic force applied to the damped mass oscillator
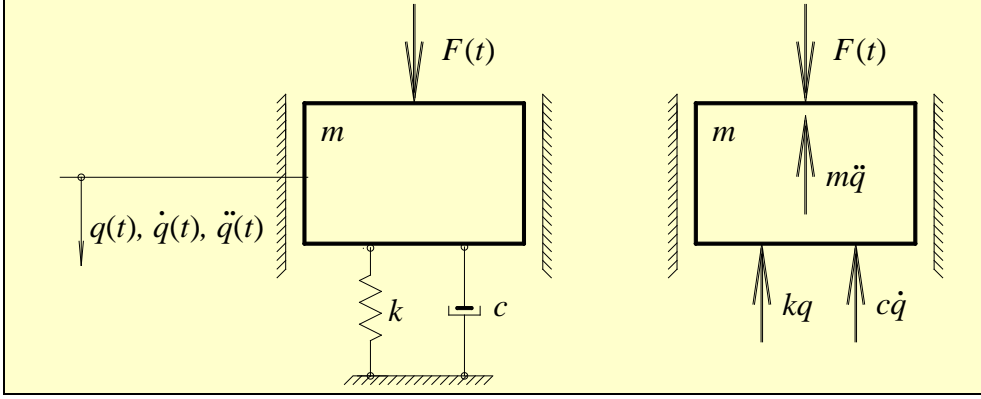

Fig. 2.11 Single-degree-of-freedom system with viscous damping, excited by a harmonic force acting on a mass

where

$$F(t) = F_S \sin \omega t + F_C \cos \omega t = F_o \sin(\omega t + \varphi) \qquad (2.38)$$

$$F_o = \operatorname{am} F(t) = \sqrt{F_S^2 + F_C^2}, \qquad \varphi = \operatorname{arc} \operatorname{tg}(F_C/F_S) \qquad (2.39)$$

The differential equation of motion of a single-degree-of-freedom system with viscous damping excited by a harmonic force has the form

$$m\ddot{q} + c\,\dot{q} + k\,q = F_S \sin \omega t + F_C \cos \omega t \qquad (2.40)$$

or

$$\ddot{q} + 2\alpha\omega_n\,\dot{q} + \omega_n^2\,q = \frac{\omega_n^2}{k}(F_S \sin \omega t + F_C \cos \omega t) \qquad (2.41)$$

The total solution of Eq. (2.40) or Eq. (2.41) with harmonic force vector defined by Eq. (2.38) consist of the complementary solution and the particular solution. The complementary solution is defined by the Eq. (2.30) for the underdamped case. The particular solution of the Eq. (2.40) or Eq. (2.41) can also be found in a harmonic form

$$q(t) = q_S \sin \omega t + q_C \cos \omega t = q_o \sin(\omega t + \varphi) \qquad (2.42)$$

Substituting Eq. (2.42) into Eq. (2.40) one can achieve a set of algebraic equations

$$
\begin{aligned}
(\omega_n^2 - \omega^2)q_S - 2\alpha\omega_n\omega q_C &= F_S\omega_n^2/k \\
2\alpha\omega_n\omega q_S + (\omega_n^2 - \omega^2)q_C &= F_C\omega_n^2/k
\end{aligned}
\qquad (2.43)
$$

By defining the frequency ratio

$$\eta = \omega / \omega_n \tag{2.44}$$

Eqs. (2.43) can be written down as

$$
\begin{aligned}
(1-\eta^2)q_S - 2\alpha\eta q_C = F_S/k \\
2\alpha\eta q_S + (1-\eta^2)q_C = F_C/k
\end{aligned}
\tag{2.45}
$$

The solution of the Eq. (2.45) is

$$
\begin{aligned}
q_S = h_1 F_S + h_2 F_C, \\
q_C = h_1 F_C - h_2 F_S,
\end{aligned}
\tag{2.46}
$$

where

$$h_1 = \frac{1}{k}\frac{1-\eta^2}{(1-\eta^2)^2+(2\alpha\eta)^2} \qquad h_2 = \frac{1}{k}\frac{2\alpha\eta}{(1-\eta^2)^2+(2\alpha\eta)^2} \tag{2.47}$$

The particular solution of Eq. (2.40) which is the steady-state response of the system, Fig. 2.10, can be finally written in form

$$q(t) = \frac{v_d}{k}F_S\sin(\omega t - \psi) + \frac{v_d}{k}F_C\cos(\omega t - \psi) \tag{2.48}$$

and

$$\text{am } q = \frac{v_d}{k}\sqrt{F_S + F_C} = v_d\frac{F_o}{k} = v_d\, q_{st} \tag{2.49}$$

where the ratio of a steady-state amplitude of $q(t)$ to the static deflection q_{st} is defined as the dynamic magnification factor described by formula

$$v_d = \frac{\text{am } q}{q_{st}} = \frac{1}{\sqrt{(1-\eta^2)^2+(2\alpha\eta)^2}} \qquad \text{and} \qquad \psi = \arctan\frac{2\alpha\eta}{1-\eta^2} \tag{2.50}$$

By combining the complementary solution, or transient response, from Eq. (2.30) with the particular solution, or steady-state response, from Eq. (2.48), the total solution of Eq. (2.40) or Eq. (2.41) can be arrived at:

$$q(t) = e^{-\alpha\omega_n t}(q_S\sin\omega_d t + q_C\cos\omega_d t) + \frac{v_d}{k}F_S\sin(\omega t - \psi) + \frac{v_d}{k}F_C\cos(\omega t - \psi) \tag{2.51}$$

It must be noted that the constants of integration q_S and q_C should be calculated from the initial conditions with the use of the total response given by Eq. (2.51) and not from just the transient component of response given in Eq. (2.30). As the exponential factor $e^{-\alpha\omega_n t}$ will always be present in a damped system, the transient component of response disappears, and only the steady-state motion given by Eq. (2.48) remains.

From Eqs. (2.48) it is seen that the steady-state response of the system is a time history of the force with phase delay. The variation of the steady-state displacement am q/q_{st} (dynamic amplification factor v_d) as a function of the frequency ratio η and the damping ratio α is shown in Fig. 2.12.
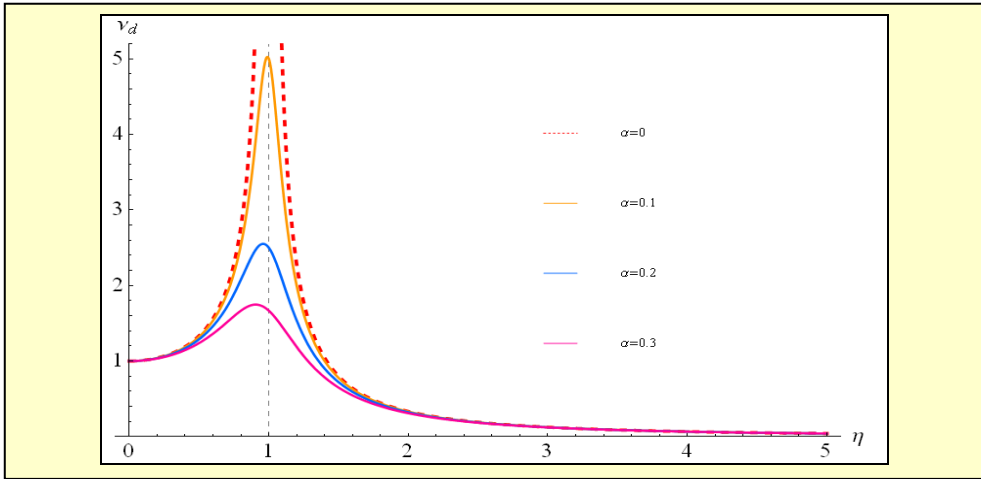

Fig. 2.12 Dynamic magnification factor v_d as a function of frequency ratio for various amounts of damping

Analogically to the magnification factor, the phase angle delay ψ, given by Eq. (2.51) and shown in Fig. 2.13, varies with the frequency ratio η and the damping ratio α. When $\eta < 1$, it is the case of high tuning, i.e. structure parameters cause a greater value of natural frequency which remains greater than the frequency of excitation; when $\eta > 1$, it is the case of low tuning, i.e. structure parameters cause the opposite situation.
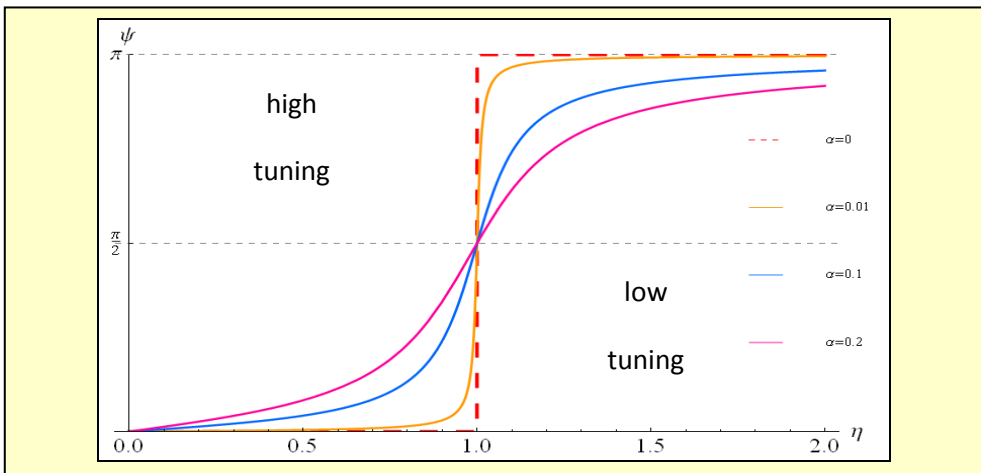

Fig. 2.13 Phase angle delay ψ as a function of frequency ratio for various damping values

It can be noted in Fig. 2.12 that the peak amplitude occurs at the frequency ratio

$$\eta = \sqrt{1-2\alpha^2} < 1 \qquad (2.52)$$

and the exact maximum value of dynamic amplification factor ν_d is (valid for $\alpha < 1$)

$$\nu_{d\max} = \frac{1}{2\alpha\sqrt{1-\alpha^2}} \qquad (2.53)$$

For a lightly damped system $\alpha \ll 1$, the peak amplitude occurs at a frequency ratio very close to $\eta = 1$ (resonance). In practice, then, the maximum value of the dynamic amplification factor $\nu_{d\max}$ may be assumed to occur at resonance, which means that the dynamic magnification factor has the maximum value

$$\nu_{d\max} \approx \nu_r = \frac{1}{2\alpha} \qquad (2.54)$$

It can be also seen from Eq. (2.54) that at the resonance ($\eta = 1$) the dynamic magnification factor is inversely proportional to the damping ratio.
From Eq. (2.50) it follows that for undamped systems ($\alpha = 0$) the amplification factor takes the form

$$\nu_d = \frac{1}{\left|1-\eta^2\right|} \qquad (2.55)$$

From the expression in Eq. (2.55) it follows that the maximum value of the amplification factor for an undamped system is equal to infinity, (Fig. 2.5).
Velocity and acceleration response can be achieved by differentiating Eq. (2.48)

$$\dot{q}(t) = \omega \frac{\nu_d}{k} F_S \cos(\omega t - \psi) - \omega \frac{\nu_d}{k} F_C \sin(\omega t - \psi)$$
$$= \omega \frac{\nu_d}{k} F_S \sin(\omega t - \theta) + \omega \frac{\nu_d}{k} F_C \cos(\omega t - \theta) \qquad (2.56)$$

where angle

$$\theta = \psi + \frac{\pi}{2} \qquad (2.57)$$

is the phase shift between the velocity and the exciting force. From the comparison of formulas Eq. (2.56) and Eq. (2.48), one can notice that the velocity and the displacement are $\pi/2$ out of phase from each other.

The amplitude of the velocity is

$$\text{am } \dot{q} = \omega \text{ am } q = \eta \nu_d \frac{F_o}{\sqrt{km}} = \nu_v \frac{F_o}{\sqrt{km}} \qquad (2.58)$$

where a velocity response factor is described by formula

$$v_v = \frac{\eta}{\sqrt{(1-\eta^2)^2 + (2\alpha\eta)^2}}$$

(2.59)

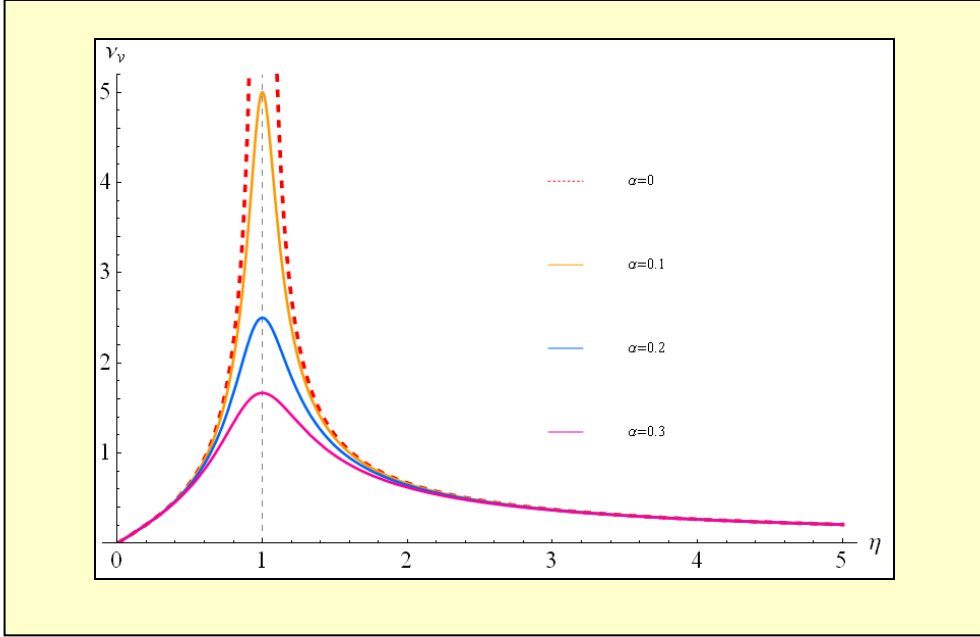The velocity response factor is shown in Fig. 2.14.

Fig. 2.14 The velocity response factor v_v as a function of frequency ratio for various amounts of damping

The acceleration response is obtained by differentiating Eq. (2.56)

$$\ddot{q}(t) = -\omega^2 \frac{v_d}{k} F_S \sin(\omega t - \psi) - \omega^2 \frac{v_d}{k} F_C \cos(\omega t - \psi) =$$
$$= \omega^2 \frac{v_d}{k} F_S \sin(\omega t - \beta) + \omega^2 \frac{v_d}{k} F_C \cos(\omega t - \beta)$$

(2.60)

where angle

$$\beta = \psi + \pi$$

(2.61)

is the phase shift between the velocity and the exciting force. It is interesting to note that from the comparison of formulas Eq. (2.50), Eq. (2.56) and Eq. (2.48), the acceleration and the displacement are π out of phase from each other, and that the velocity and the acceleration as well as the velocity and the displacement are $\pi/2$ out of phase from each other. The amplitude of the acceleration is

$$\text{am } \ddot{q} = \omega^2 \text{ am } q = \eta^2 \nu_d \frac{F_o}{m} = \nu_a \frac{F_o}{m} \qquad (2.62)$$

where an acceleration response factor is described by formula

$$\nu_a = \frac{\eta^2}{\sqrt{(1-\eta^2)^2 + (2\alpha\eta)^2}} \qquad (2.63)$$

The acceleration response factor is shown in Fig. 2.15.
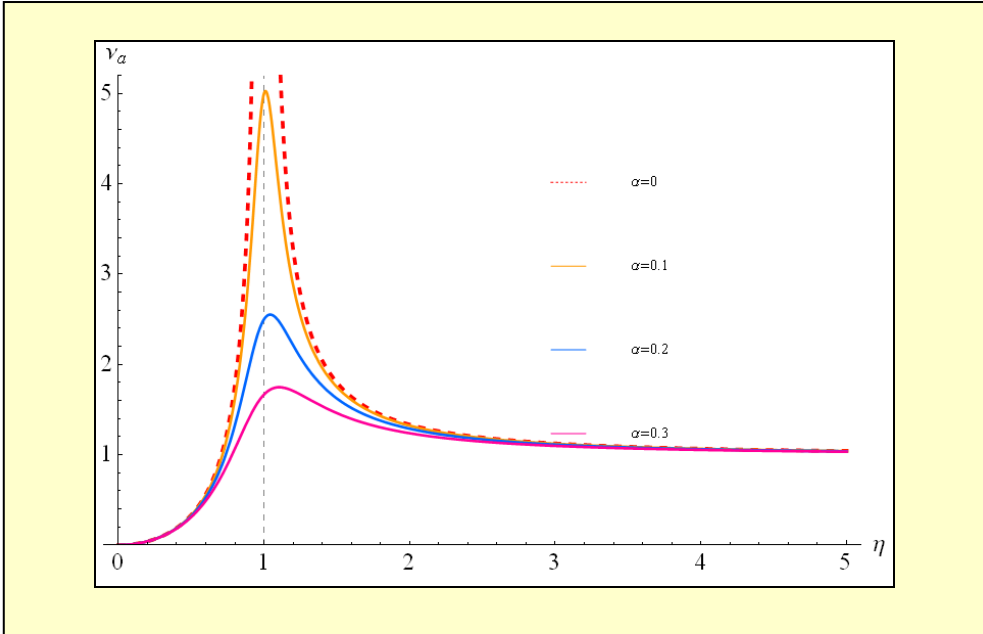

Fig. 2.15 The acceleration response factor ν_a as a function of frequency ratio for various amounts of damping

The graphs drawn for displacement (Fig. 2.12), velocity (Fig. 2.14) and acceleration (Fig. 2.15) differ distinctly.

Conclusions:

- The damping ratio has a very large influence on the amplitude and phase angle delay in the frequency region near resonance when $\eta = \omega/\omega_n \approx 1$.

- The maximum magnification factor occurs for $\eta = \omega/\omega_n \approx 1 < 1$.

- When the damping is small, $\alpha \ll 1$, the amplitude and the phase are almost independent of α. The damping and inertia forces are then very small, so that the exciting force is almost equal in value to the spring force. The spring force and inertia force are almost balanced (are counterbalanced).

- When $\eta = \omega/\omega_n \approx 1$ the damping force and the exciting force are almost equal. The spring force and the inertia force are almost balanced.

- When the frequency ratio $\eta = \omega/\omega_n \gg 1$, the phase angle ψ approaches π. The exciting force then almost equals the inertia force.

- The amplitude approaches the static displacement when $\eta = \omega/\omega_n \approx 0 \ll 1$.

- The amplitude of displacement becomes small when $\eta = \omega/\omega_n \gg 1$.

- The phase angle is very sensitive to the ratio $\eta = \omega/\omega_n$ in the region of near-resonance for small damping.

- When $\eta = \omega/\omega_n = \sqrt{1 - 2\alpha^2}$, then the dynamic amplification factor reaches a maximum $v_{d\max} = 1/2\alpha\sqrt{1-\alpha^2}$.

- The amplitude of resonance is found to be am $q = \dfrac{q_{st}}{2\alpha}$.

- The velocity response factor approaches zero $(v_v = 0)$ as $\eta = \omega/\omega_n = 0$ or $\eta = \omega/\omega_n \to \infty$.

- The acceleration response factor approaches zero $(v_a = 0)$ as $\eta = \omega/\omega_n = 0$ and approaches unity $(v_a = 1)$ as $\eta = \omega/\omega_n \to \infty$.

- The displacement and the velocity are $\pi/2$ out of phase.

- The velocity and the acceleration and are $\pi/2$ out of phase.

- The displacement and the acceleration are π out of phase.

2.5.4.2. Force Transmission to Foundation

The force transmitted to the foundation of the system is

$$F_T = F_d + F_k = c\dot{q} + kq \tag{2.64}$$

Substituting Eq. (2.48) and Eq. (2.56) into Eq. (2.64) yields the steady-state solution

$$
\begin{aligned}
F_T &= v_d(F_C + 2\alpha\eta F_S)\cos(\omega t - \psi) + v_d(F_S - 2\alpha\eta v F_C)\sin(\omega t - \psi) = \\
&= v_d\sqrt{1 + (2\alpha\eta)^2}\, F_o \sin(\omega t - \psi + \phi) = v_T F_o \sin(\omega t - \Omega) = \\
&= \mathrm{am}\, F_T \sin(\omega t - \Omega)
\end{aligned}
\tag{2.65}
$$

in which the transmissibility v_T defined as the ratio between the amplitude of the force transmitted to the foundation and the amplitude of the applied force (see Fig. 2.16) is

$$v_T = \frac{\mathrm{am}\, F_T}{F_o} = v_d\sqrt{1 + (2\alpha\eta)^2} = \sqrt{\frac{1 + (2\alpha\eta)^2}{(1 - \eta^2)^2 + (2\alpha\eta)^2}} \tag{2.66}$$

where

$$\Omega = \psi - \phi \tag{2.67}$$

$$\phi = \arctan\frac{F_c + 2\alpha\eta F_s}{F_s - 2\alpha\eta v F_c} \tag{2.68}$$

Since the displacement and the velocity are $\pi/2$ out of phase from each other, the forces $F_d = c\dot{q}$ and $F_k = kq$ are also $\pi/2$ out of phase from each other, and the magnitude of the transmitted force can be achieved from

$$\mathrm{am}\, F_T = \sqrt{(c\,\mathrm{am}\,\dot{q})^2 + (k\,\mathrm{am}\,q)^2} \tag{2.69}$$

By substituting Eq. (2.49) and Eq. (2.58) into Eq. (2.69) one can obtain the same formula which would follow from Eq. (2.66), that is

$$\mathrm{am}\, F_T = v_d\sqrt{1 + (2\alpha\eta)^2}\, F_o = v_T F_o \tag{2.70}$$
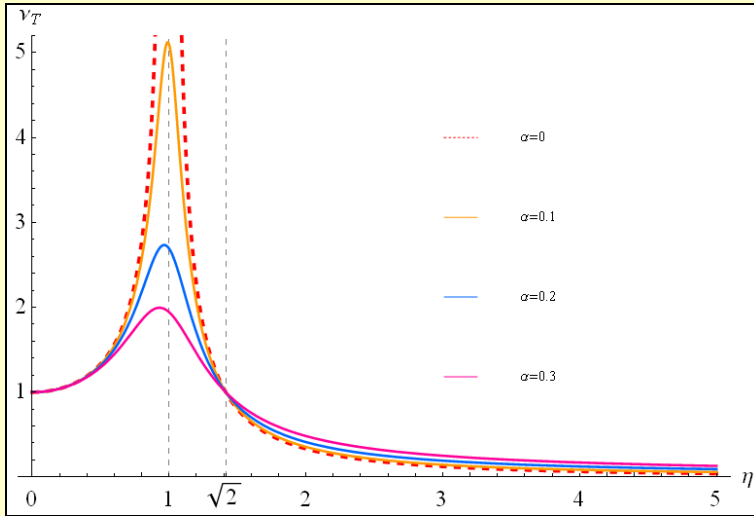
Fig. 2.16 The transmissibility ν_T as a function of frequency ratio for various amounts of damping

2.5.4.3. Resonance Frequencies

For an SDOF system undergoing harmonic excitation, displacement, velocity, and acceleration have **three** different resonance frequencies (points where they reach their maxima) which can be expressed in terms of their relation to the undamped natural frequency ω_n.

Conclusions:

For an SDOF system undergoing harmonic excitation, displacement, velocity, and acceleration have **three** different resonance frequencies

- Displacement resonance frequency: $\omega_n\sqrt{1-2\alpha^2}$

- Velocity resonance frequency: ω_n

- Acceleration resonance frequency: $\omega_n\big/\sqrt{1-2\alpha^2}$

It can be seen that these frequencies differ only slightly. Because of this, especially for the small degree of damping which is characteristic of civil engineering structures, these frequencies may in practice be assumed to be the same, and equal to the natural frequency.

2.5.4.4. Vibration Due to a Rotating Eccentric Weight
(Inertial Excitation)

In the mass-spring-damper system shown in Fig. 2.17, an unbalanced mass m_r, rotating with the angular frequency ω, is fixed to the mass m in a way that allows for rotating movement only. The mass m_r follows a circular path of radius e (eccentricity radius) with respect to the pivot.
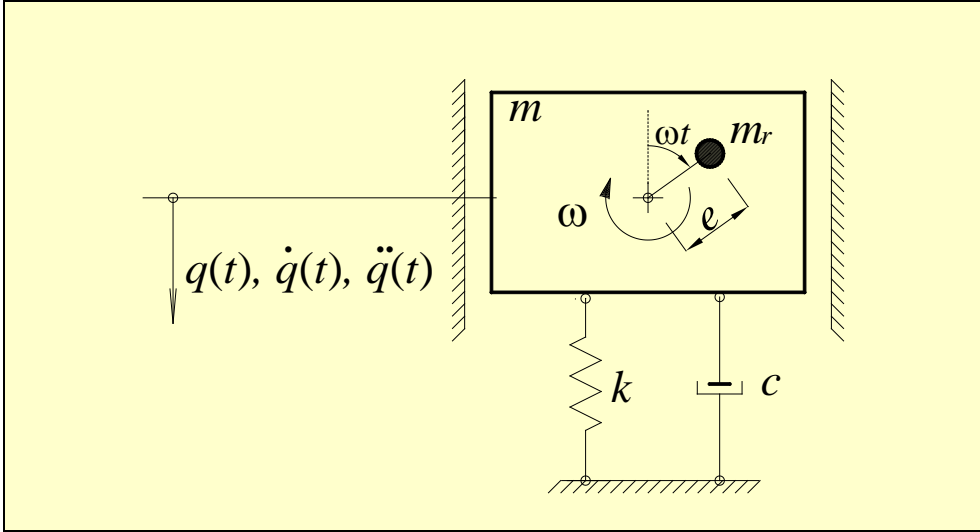

Fig. 2.17 Single DOF system with viscous damper, excited by rotating eccentric weight.

The centrifugal force acting on the mass is described by formula

$$F_o = M_o \omega^2 = m_r e \omega^2 \qquad (2.71)$$

where

$$\mathfrak{I}_o = m_r e \qquad (2.72)$$

is the moment of unbalance. It must be underlined that the amplitude of the exciting force is proportional to the square of the angular velocity (frequency). The projection of the centrifugal force onto the direction of oscillation is

$$F(t) = F_o \cos \omega t = m_r e \omega^2 \cos \omega t \qquad (2.73)$$

This force ought to be substituted with the right side of Eq. (2.40) and the equation of motion can be written in form

$$m\ddot{q} + c\dot{q} + k q = m_r e \omega^2 \cos \omega t \qquad (2.74)$$

Now, with accordance to Eq. (2.49), one can achieve

$$\text{am } q = v_d \frac{F_o}{k} = v_d \frac{m_r e \omega^2}{k} = \eta^2 v_d \frac{m_r e}{m} \tag{2.75}$$

The magnitude of the force transmitted to the foundation, in accordance with Eq. (2.71)

$$\text{am } F_T = v_d \sqrt{1+(2\alpha\eta)^2} F_o = \eta^2 v_T m_r e \omega_n^2 \tag{2.76}$$

It can be seen that the dynamic magnification factor, Fig. 2.18, and the transmissibility, Fig. 2.19, are now described by new formulas, respectively

$$v_r = \eta^2 v_d = \frac{\eta^2}{\sqrt{(1-\eta^2)^2 + (2\alpha\eta)^2}} \tag{2.77}$$

$$v_T' = \eta^2 v_T = \eta^2 \sqrt{\frac{1+(2\alpha\eta)^2}{(1-\eta^2)^2 + (2\alpha\eta)^2}} \tag{2.78}$$
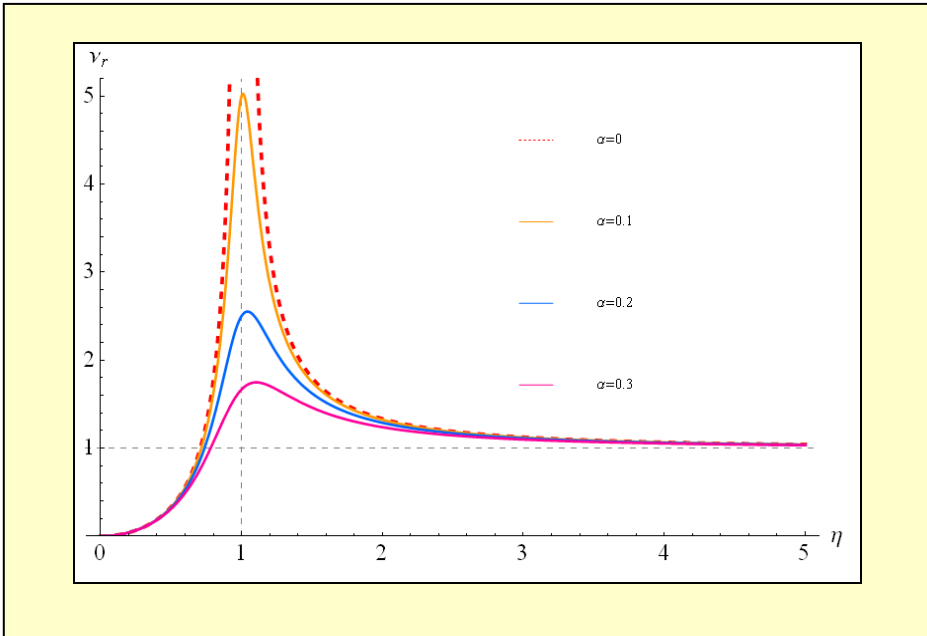

Fig. 2.18 Dynamic magnification factor v_r as a function of frequency ratio for various amounts of damping
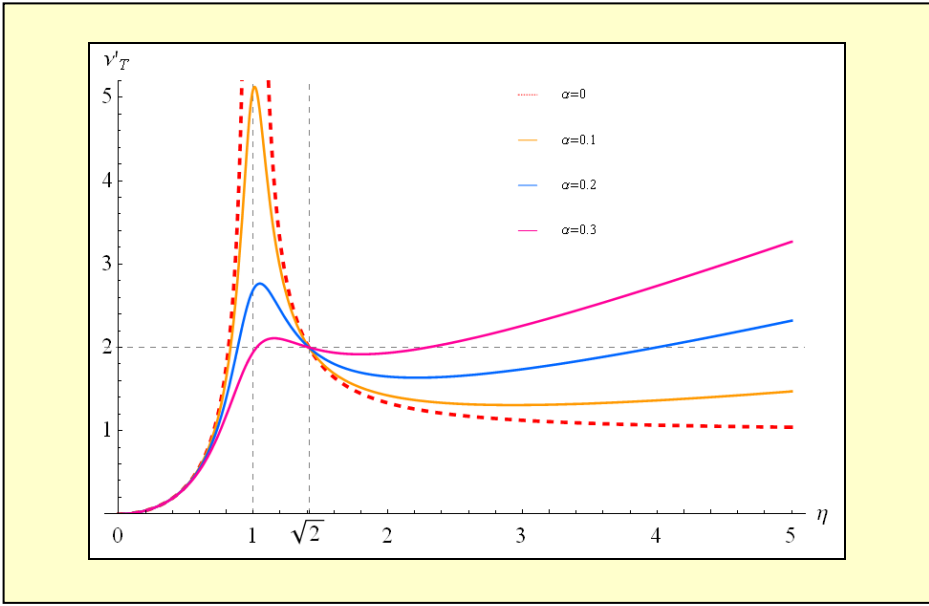
Fig. 2.19 The transmissibility v'_T as a function of frequency ratio for various amounts of damping

2.5.4.5. Kinematically Forced Vibration

There are many cases where the foundation or support of a structure undergoes motion which varies in time. The movement of the foundations may have to be considered in the analysis of the dynamic response of structures subjected to ground motion by seismic (earthquakes) or paraseismic excitation (mining tremors, the dynamic action of machines). A single-degree-of-freedom system with a viscous damper, excited in forced vibration by foundation motion $u(t)$, is shown in Fig. 2.20.
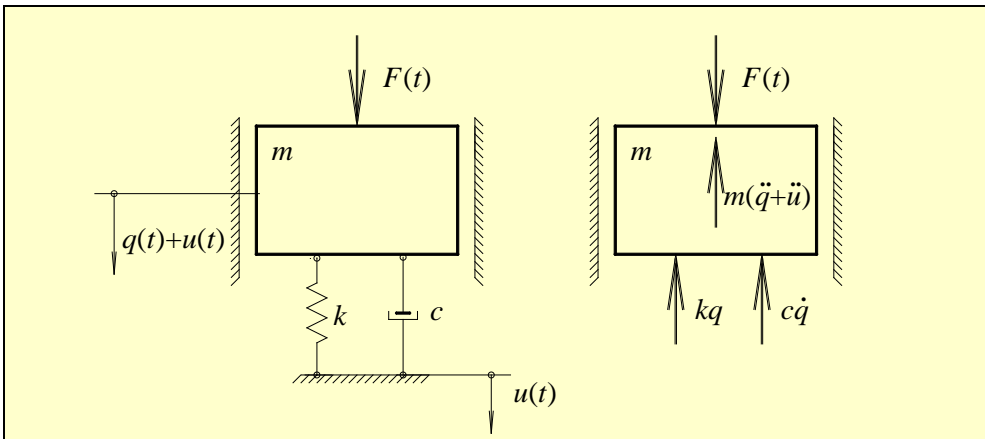

Fig. 2.20 Single-degree-of-freedom system with viscous damper, excited in forced vibration by foundation motion

The motion of mass m is described by a sum of the foundation motion $u(t)$ and the relative motion $q(t)$ between the mass m and the support, i.e.

$$z(t) = u(t) + q(t) \tag{2.79}$$

The equation of motion, then, can be written in form

$$m(\ddot{u} + \ddot{q}) + c\,\dot{q} + k\,q = 0 \tag{2.80}$$

or in a more common way

$$m\ddot{q} + c\,\dot{q} + k\,q = -m\ddot{u} \tag{2.81}$$

The factor on the right side of Eq. (2.81) has the same function as the excitation force in Eq. (2.40). If the function $u(t)$ is known, the right side of Eq. (2.81) is definite, then for some types of this function (especially if it is harmonic) this equation can be solved analytically. Let us assume that $u(t)$ is a harmonic function

$$u(t) = u_o \sin \omega t \tag{2.82}$$

After two times differentiation of the function in Eq. (2.82), it can be substituted to Eq. (2.81), and one can achieve

$$m\ddot{q} + c\,\dot{q} + k\,q = m\omega^2 u_o \sin \omega t \tag{2.83}$$

As has been stated earlier, the right hand side of the Eq. (2.83) may be interpreted as a force (equivalent acting force). The acting force can therefore be written in form

$$F(t) = F_o \sin \omega t \tag{2.84}$$

where

$$F_o = m u_o \omega^2 \tag{2.85}$$

The amplitude of the exciting force is then again proportional to the square of the angular velocity (frequency), like in the case of the eccentric rotating mass – see Eq. (2.71).

Now, with accordance to Eqs. (2.49) and (2.75), one can achieve

$$q(t) = amq \sin(\omega t - \psi) \tag{2.86}$$

where

$$\text{am } q = v_d \frac{F_o}{k} = v_d \frac{mu_o \omega^2}{k} = \eta^2 v_d u_o = v_r u_o \qquad (2.87)$$

The magnitude of the force transmitted to the foundation, in accordance with Eq. (2.71)

$$\text{am } F_T = v_d \sqrt{1+(2\alpha\eta)^2}\, F_o = v_T\, \eta^2 k u_o = v_T'\, k u_o \qquad (2.88)$$

It can be seen that the dynamic magnification factor and transmissibility (Eqs. (2.87) and (2.88)) are now described by the same formulas as in the case of the eccentric rotating mass – see Eq. (2.77) and (2.78), respectively.

The total displacement of the mass m is

$$z(t) = u(t) + q(t) = u_o \sin \omega t + v_r\, u_o \sin(\omega t - \psi) \qquad (2.89)$$

The amplitude of displacement of mass m is

$$\text{am } z(t) = u_o \sqrt{1+v_r^2 + 2v_r \cos\psi} \qquad (2.90)$$

Since

$$\cos\psi = \frac{1}{\sqrt{1+\tan^2\psi}} = (1-\eta^2)v_d \qquad (2.91)$$

using definition of v_d, Eq. (2.50), the final formula, of the total displacement of the mass m is

$$\text{am } z(t) = v_d \sqrt{1+(2\alpha\eta)^2}\, u_o = v_T u_o \qquad (2.92)$$

It ought to be underlined that exactly the same function describes the transmissibility of motion (from the foundation to the structure, Eq. (2.92)), and of force (from the structure to the foundation, Eq. (2.66)).

It is interesting to note that, if $\omega \gg \omega_n$, $v_r = \eta^2 v_d \rightarrow 1$, $\psi = \pi$, $v_T \rightarrow 0$, and then

$$q(t) = -u_o \sin \omega t = -u(t) \qquad \text{and} \qquad z(t) = u(t) + q(t) = 0 \qquad (2.93)$$

The mass remains at rest in an external inertial frame of reference (x, y) (the observer is situated neither in the system nor on the ground, but is outside of the system), and its relative movement in a non-inertial frame of reference is connected with the ground which moves with accordance to the $z(t)$ function. The relative motion of the mass reflects in antiphase ($q(t) = -u(t)$) the kinematic excitation caused by the movement of the ground.

Conclusions:

In case of inertial and kinematical excitation:

- The amplitudes of displacement and force transmitted to the foundation are described now by new functions:
  - dynamic magnification factor $v_r = \eta^2 v_d$ instead the v_d
  - transmissibility $v'_T = \eta^2 v_T$ instead the v'_T
- The important features distinguish these new functions with comparison to the first ones:
  - if $\eta = \omega/\omega_n \to 0$ the values of functions approach zero ($v_r \to 0$) instead approaching a limit one ($v_d \to 1$)
  - if $\eta = \omega/\omega_n \to \infty$ the value of the v_r approaches a limit one ($v_r \to 1$) instead approaching zero ($v_d \to 0$)
  - if $\eta = \omega/\omega_n \to \infty$ also v'_T approaches infinity ($v'_T \to \infty$) asymptotically to the line $2\alpha\eta$;
    - for low tuning the response of the system can be greater than in the resonance (see Fig. 2.19)
    - for greater values of damping ratio α the values of the transmissibility v'_T grow faster
- The damping ratio still has a very large influence on the amplitude and phase angle delay in the frequency region near resonance when $\eta = \omega/\omega_n \approx 1$.
- The phase angle is very sensitive to the ratio $\eta = \omega/\omega_n$ in the region of near-resonance for small damping.
- The maximum magnification factor occurs now for $\eta = \omega/\omega_n \to \infty$.

For kinematically forced vibration

- When the frequency ratio $\eta = \omega/\omega_n \gg 1$, dynamic magnification factor $v_r = \eta^2 v_d \to 1$, the transmissibility $v_T \to 0$ the phase angle delay ψ approaches π.
  - The mass remains at rest in an external inertial frame of reference $\boxed{z(t) = u(t) + q(t) = 0}$
  - The relative motion of the mass reflects in antiphase the kinematic excitation caused by the movement of the ground $\boxed{q(t) = -u_o \sin \omega t = -u(t)}$

2.5.5. Evaluation of Equivalent Viscous Damping

In structures subject to significant or strong vibration, such as the ones excited by earthquakes or strong winds, special devices are usually installed to introduce viscous damping into the system. In standard structures such devices are not used. Even so, additional damping is assumed to occur in such structures as well, as it allows the differential equation of motion to remain linear for damped dynamic systems vibrating in the elastic range. The damping assumed is referred to as equivalent viscous damping, and the numerical value of its coefficient is usually calculated on the basis of experimental measurements using a harmonic force test. A different way to determine the damping in a system is to specify the so called logarithmic decrement of damping in a free oscillation test.

2.5.5.1. Logarithmic Decrement

The definition of the logarithmic decrement has been presented in Chapter 2.5.3.4. Another way to establish the logarithmic decrement involves calculating the ratio of two consecutive peak accelerations on a free oscillation (acceleration) curve. As it is far easier to experimentally measure accelerations than displacements, this modification is of great practical importance. The Eq. (2.31) can be in form

$$q(t) = Ce^{-\alpha\omega_n t}\cos(\omega_d t - \gamma) \tag{2.94}$$

where constant C is

$$C = \sqrt{q_o^2 + \frac{(\dot{q}_o + q_o\alpha\omega_n)^2}{\omega_d^2}} \tag{2.95}$$

$$\tan\gamma = \frac{\dot{q}_o + q_o\alpha\omega_n}{q_o\omega_d} \tag{2.96}$$

Differentiating Eq. (2.94) two times leads to formula

$$\ddot{q}(t) = Ce^{-\alpha\omega_n t}\{[-\alpha\omega_n\cos(\omega_d t - \gamma) - \omega_d\sin(\omega_d t - \gamma)](-\alpha\omega_n) + \\ + [\alpha\omega_n\omega_d\sin(\omega_d t - \gamma) - \omega_d^2\cos(\omega_d t - \gamma)]\} \tag{2.97}$$

At time t_1, when $\cos(\omega_d t_1 - \gamma) = 1$ and $\sin(\omega_d t_1 - \gamma) = 0$

$$\ddot{q}(t_1) = Ce^{-\alpha\omega_n t_1}(\alpha^2\omega_n^2 - \omega_d^2) \tag{2.98}$$

and at time $t_2 = t_1 + T_d$, corresponding to a period later, when again $\cos(\omega_d t_2 - \gamma) = 1$ and $\sin(\omega_d t_2 - \gamma) = 0$,

$$\ddot{q}(t_2) = Ce^{-\alpha\omega_n(t_1+T_d)}(\alpha^2\omega_n^2 - \omega_d^2) \tag{2.99}$$

The ratio of acceleration at time t_1 and t_2 is then

$$\frac{\ddot{q}(t_1)}{\ddot{q}(t_2)} = e^{\alpha \omega_n T_d} \qquad (2.100)$$

After taking the natural logarithmic of the ratio Eq. (2.100), the result is the logarithmic decrement in terms of the acceleration

$$\vartheta = \ln \frac{\ddot{q}(t_1)}{\ddot{q}(t_2)} = \alpha \omega_n T_d \qquad (2.101)$$

which is identical to the expression for the logarithmic decrement given by Eq. (2.35) in terms of displacement. From Eq. (2.35) or Eq. (2.101) the result Eq. (2.37) describing equivalent viscous damping ratio is

$$\alpha_{eq} = \frac{\vartheta}{2\pi} \qquad (2.102)$$

2.5.5.2. Evaluation of Damping at Resonance

It can be seen in Chapter 2.5.3 that, thanks to the free-vibration decay curve for SDOF systems and to the use of Eqs. (2.100) or (2.35), the damping can be evaluated by calculating the logarithmic decrement. A different way to calculate damping is to apply steady-state harmonic response in a range of frequencies in the neighborhood of resonance. It is enough to plot a frequency response curve for harmonic excitation in this range of frequencies, Fig. 2.21.
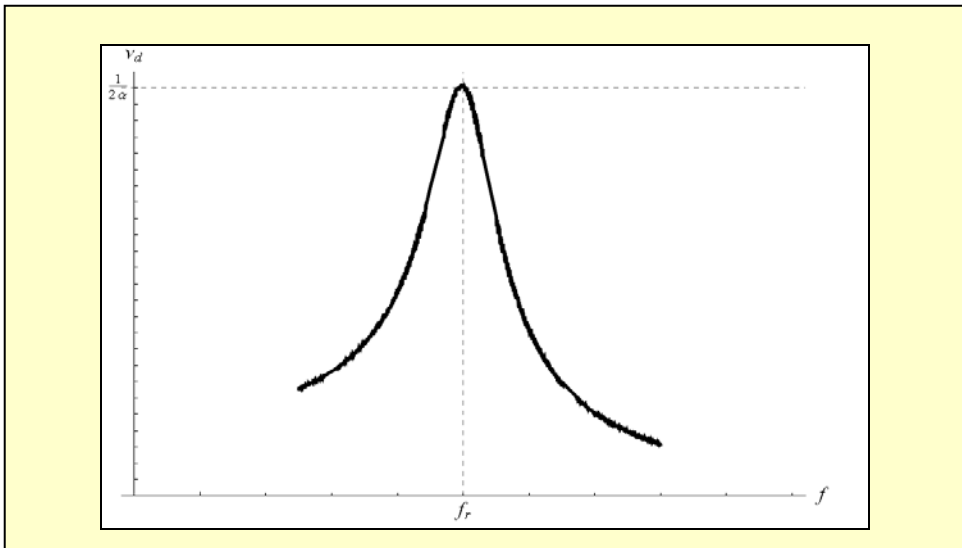

Fig. 2.21 Experimental frequency response curve for damped system in the neighborhood of resonance

It is seen from Eq. (2.55) that the damping ratio is given by

$$\alpha = \frac{1}{2v_r}$$

(2.103)

where $v_r = v_d(\eta = 1)$ is the dynamic magnification factor evaluated at resonance.

In practice, the equivalent damping ratio α_{eq} is determined from the dynamic magnification factor evaluated by measuring the static displacement q_{st} and the maximum amplitude $q_m = \max(\text{am } q)$ during the harmonic excitation of the structure with the resonant frequency and by measuring

$$v_{d\max} = \frac{q_m}{q_{st}}$$

(2.104)

Then, the equivalent damping ratio α_{eq} is

$$\alpha_{eq} = \frac{q_{st}}{2q_m}$$

(2.105)

In ordinary structures, the error involved in evaluating the equivalent damping ratio α_{eq} using the approximate Eq. (2.104) is not significant. Calculating the static displacement necessary in the formula in Eq. (2.105) may present a considerable problem, however. This is due to the fact that it is sometimes difficult to apply a force of proper value to the structure in the required place and direction.

2.5.5.3. Hysteresis

The most common method of evaluating equivalent viscous damping is to equate the energy dissipated in the period of vibration of the actual structure to the energy dissipated in an equivalent viscous system.

When the viscous damped SDOF system shown in Fig. 2.11 undergoes steady-state forced vibration defined by Eq. (2.42), where $\varphi = 0$, the equation takes the form

$$q(t) = q_o \sin \omega t$$

(2.106)

The resultant force of the spring and damper acting on the mass, in accordance with the Eq. (2.64), is

$$F_T = c\dot{q} + kq = c\omega q_o \cos \omega t + kq_o \sin \omega t$$

(2.107)

Equations (2.107) and (2.106) define the relation between F_T and q; this relation is the ellipse shown in Fig. 2.22.
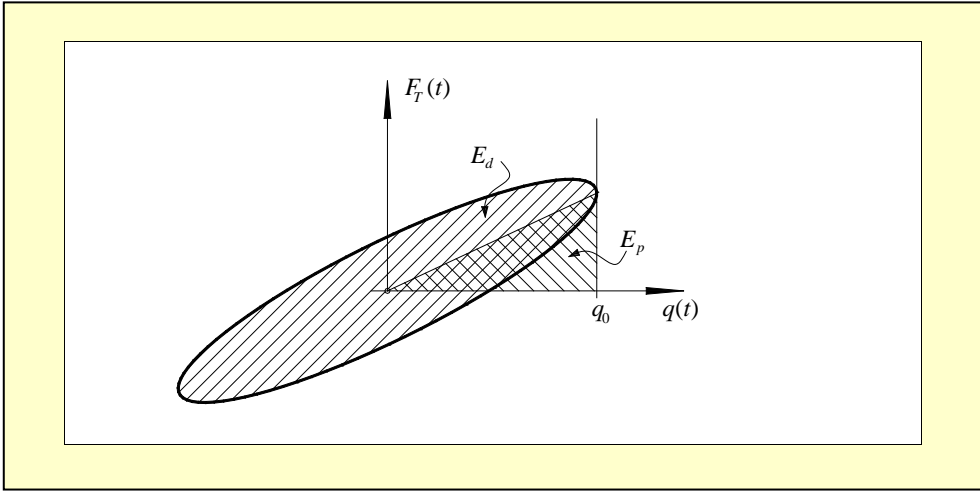
Fig. 2.22 Hysteresis curve for a spring and viscous damper in parallel

The maximum potential strain energy E_p stored at maximum displacement q_o, if the system is elastic, is given by the triangular area under the segment of the line of displacement from zero to the point of maximum displacement, marked in Fig. 2.22.

The area enclosed in the loop (the ellipse, in the given case) during one cycle of vibration is equal to the energy dissipated in one cycle of harmonic oscillations E_d. The energy dissipated by the damper is known as hysteresis loss. This energy can be calculated as a work of force F_T on displacement q, according to the formula

$$E_d = W = \int_{T}^{T+2\pi/\omega} F_T \frac{dx}{dt} dt = \pi c \omega q_o^2 = 4\pi \eta \, \alpha \, E_p \qquad (2.108)$$

From Eq. (2.108), the damping ratio is

$$\alpha = \frac{1}{4\pi\eta} \frac{E_d}{E_p} \qquad (2.109)$$

Consequently, the equivalent damping ratio could be obtained in the same way

$$\alpha_{eq} = \frac{1}{4\pi\eta} \frac{E_d^*}{E_p^*} \qquad (2.110)$$

where

E_d^* is the energy dissipated during a cycle of vibration, i.e. the area enclosed in the loop of hysteresis achieved experimentally during one cycle of vibration;

E_p^* is the potential strain energy, with $\eta = 1$ (resonant frequency), obtained from experimentally achieved resisting force-displacement plot, i.e. the triangular area under the segment of the line of displacement from zero to the point of maximum displacement q_o.

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2.5.5.4. Bandwidth (Half-Power) Method

The bandwidth is the difference between two frequencies corresponding to the same response amplitude. The bandwidth is related to the damping in the system. In the evaluation of damping, it is convenient to measure the bandwidth at the points where the frequency-amplitude curve obtained experimentally for a damped structure reaches $1/\sqrt{2}$ of its maximum (peak) value, Fig 2.23.
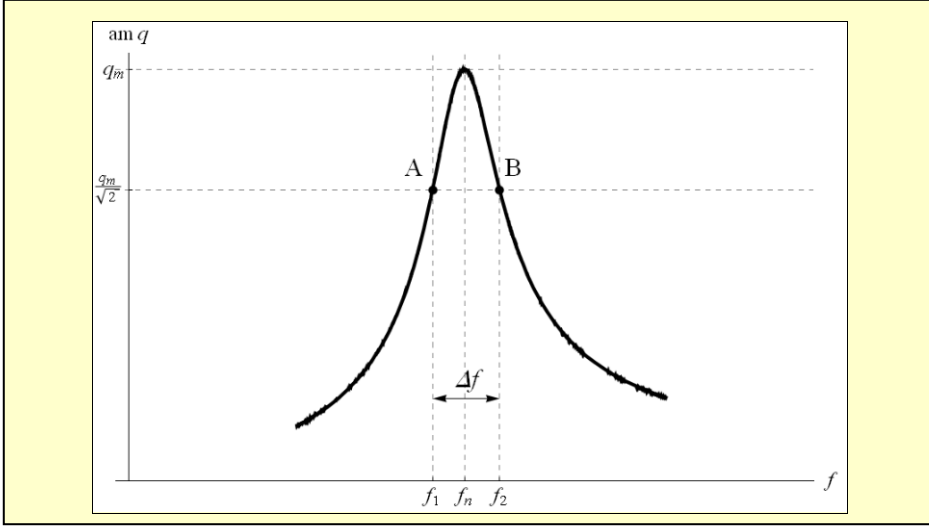

Fig. 2.23 Experimental amplitude-frequency curve

The points A and B corresponding to the frequencies f_1 and f_2 are called half-power points. In this case, then, the bandwidth is the distance between these points, i.e. $\Delta f = f_2 - f_1$. The value of frequencies for this bandwidth can also be determined by comparing the response amplitude from Eq. (2.50)

$$\text{am } q = \frac{q_{st}}{\sqrt{(1-\eta^2)^2 + (2\alpha\eta)^2}} \qquad (2.111)$$

with the $1/\sqrt{2}$ part of the maximal amplitude

$$q_m = \max(\text{am } q) = \frac{q_{st}}{2\alpha} \qquad (2.112)$$

achieved from Eq. (2.111) for the resonant frequency ($\eta = 1$). From the comparison, one can achieve the equation

$$\frac{q_{st}}{\sqrt{(1-\eta^2)^2 + (2\alpha\eta)^2}} = \frac{1}{\sqrt{2}} \frac{q_{st}}{2\alpha} \qquad (2.113)$$

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Following M. Paz and W. Leigh [2], the next mathematical operations are done as follows: "Squaring both sides and solving for the frequency ratio results in

$$\eta^2 = 1 - 2\alpha^2 \pm 2\alpha\sqrt{1 + \alpha^2}$$

(2.114)

or by neglecting α^2 in the square root term

$$\begin{aligned}
\eta_1^2 &\cong 1 - 2\alpha^2 - 2\alpha \\
\eta_2^2 &\cong 1 - 2\alpha^2 + 2\alpha \\
\eta_1 &\cong 1 - \alpha - \alpha^2 \\
\eta_2 &\cong 1 + \alpha - \alpha^2
\end{aligned}$$

(2.115)

Finally, the damping ratio is given approximately by half the difference between these half-power frequency ratios, namely

$$\alpha_{eq} = \frac{1}{2}\frac{\omega_2 - \omega_1}{\omega_n} = \frac{f_2 - f_1}{f_2 + f_1}$$

(2.116)

since

$$\frac{1}{2}\frac{\omega_2 - \omega_1}{\omega_n} = \frac{f_2 - f_1}{2f_n} \quad and \quad f_n \approx \frac{f_2 + f_1}{2} \quad . \text{"}$$

(2.117)

Finally, the equivalent damping ratio can be calculated from formula

$$\alpha_{eq} = \frac{f_2 - f_1}{f_2 + f_1}$$

(2.118)

2.5.5.5. Structural Damping

From the Eq. (2.108) it follows that hysteresis loss (the energy lost from a system due to damping) is proportional to the forcing frequency ω. On the other hand, experimental research of real civil engineering structures reveals that, in most cases, the hysteresis loss is independent of forcing frequency. To provide a better model for the damping experienced during vibration of such structures, an arbitrary term

$$c\omega_n = \gamma k$$

(2.119)

is introduced. This term, referred to as structural damping, means that the damping force is equal to the viscous damping force at some frequency, depending upon the value of damping coefficient γ, but does not vary with forcing frequency. The viscous damping coefficient calculated from Eq. (2.119) has the value

$$c = \frac{\gamma}{\omega_n}k = \kappa k$$

(2.120)

The dimensional parameter κ is the retardation time of viscoelastic solid material of Voigt-Kelvin rheological model, in which the damping effect is an inherent property of the spring material and the spring is considered to have a damper acting in parallel with it, see Fig. 2.24.

From Eq. (2.120) it follows that the damping coefficient is proportional to stiffness of the spring. After substitution of Eq. (2.120) into Eq. (1.80) the absolute value of the damping force can be now written in form

$$F_d = c\,\dot{q} = \kappa k\,\dot{q} = \kappa \dot{F}_k \tag{2.121}$$

and one can see, if stiffness coefficient is independent of time ($k \neq k(t)$), that the damping force value is proportional to the velocity of elastic reaction. In this case, the damping is a property of the system.
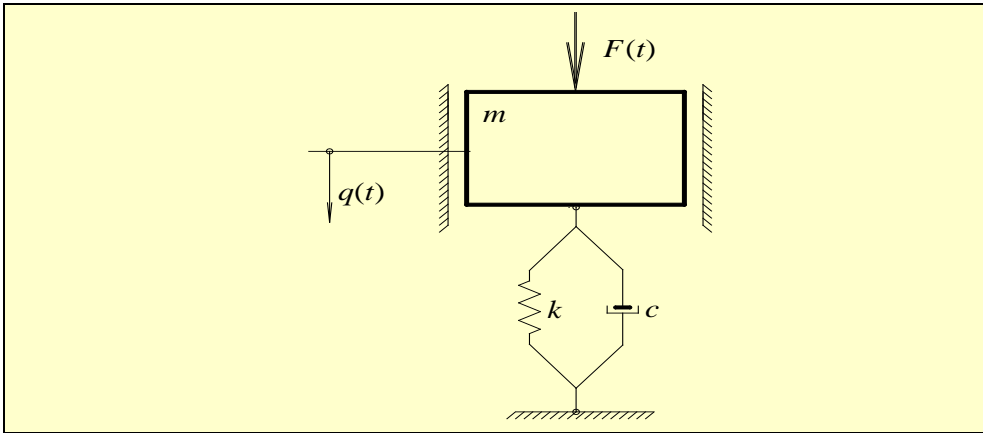

Fig. 2.24. Voigt-Kelvin rheological model

The retardation time is equal to the time needed for the function to reach the value of $1 - 1/e \approx 0.63$ of the reduced creep. It is convenient to assume that

$$\kappa = \frac{\gamma}{\omega_n} = \frac{2\alpha}{\omega_n} = \frac{2\alpha\eta}{\omega} \tag{2.122}$$

and at the resonance the dynamic amplification factor, Eqs. (2.50), (2.55) is

$$\nu_d = \nu_r = \frac{1}{\gamma} = \frac{1}{2\alpha} \tag{2.123}$$

In the case of structural damping, the set of two parametric equations describing the hysteresis curve (ellipse) can be written as

$$q(t) = q_o \sin \omega t \tag{2.124}$$

$$F_T = F_d + F_k = c\dot{q} + kq = k(\kappa\dot{q} + q) = kq_o(\gamma \cos \omega t + \sin \omega t) \tag{2.125}$$

This ellipse, which represents the relation of the damping force (transient force) F_T to the displacement q, is similar to the ellipse shown in Fig. 2.22. Therefore, the damping coefficient may be calculated in the same way as the one presented in Chapter 2.5.5.3. The equivalent damping ratio is given by

$$\alpha_{eq} = \kappa \omega_n / 2 \qquad (2.126)$$

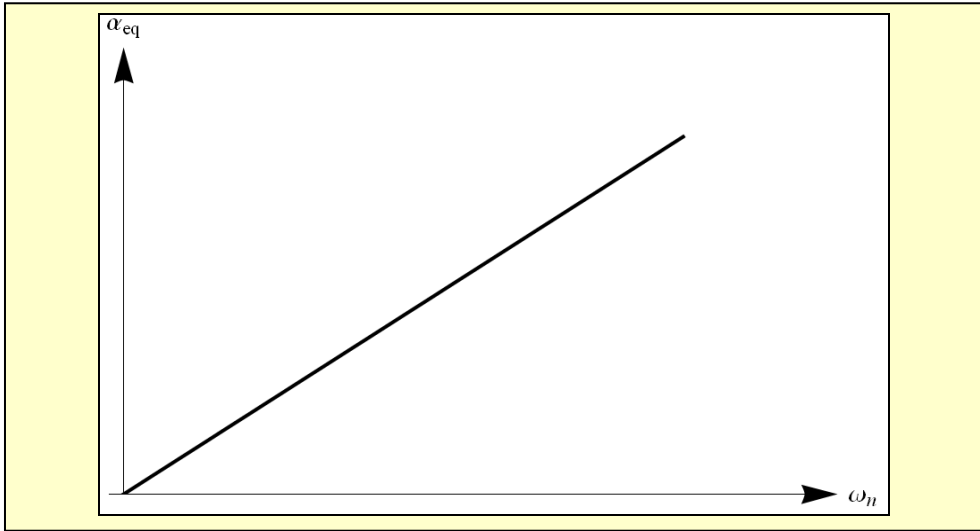The dependence of the equivalent damping ratio on the natural frequency value is shown in Fig. 2.25.

Fig. 2.25 Dependence of equivalent damping ratio on the natural frequency value

2.5.5.6. Mass damping

In this model of damping it is assumed that the damping force is proportional to the magnitude of the mass – or, to be more exact, to the momentum of the mass

$$F_d = c\,\dot{q} = \mu m\,\dot{q} \qquad (2.127)$$

so the damping coefficient is

$$c = \mu m \qquad (2.128)$$

where μ is the dimensional damping parameter. The equivalent damping ratio is given by

$$\alpha_{eq} = \frac{c}{2\sqrt{km}} = \frac{\mu m}{2\sqrt{km}} = \frac{\mu}{2\omega_n} \qquad (2.129)$$

The viscous damping, then, represents an environmental damping by external forces acting on the structure, for example the resistance of air, Fig. 2.26
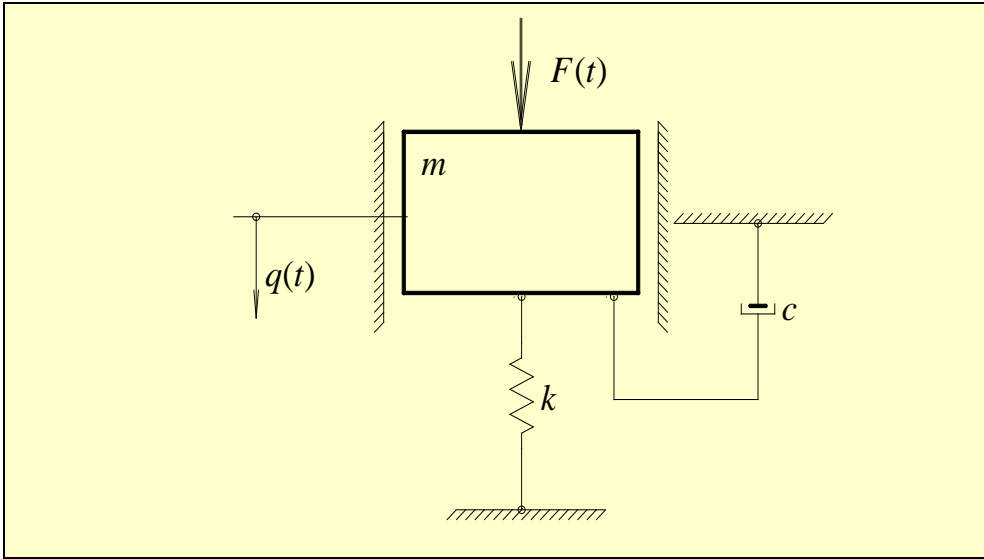
Fig. 2.26. Mass damping model

The dependence of equivalent damping ratio on the natural frequency value is shown in Fig. 2.27.
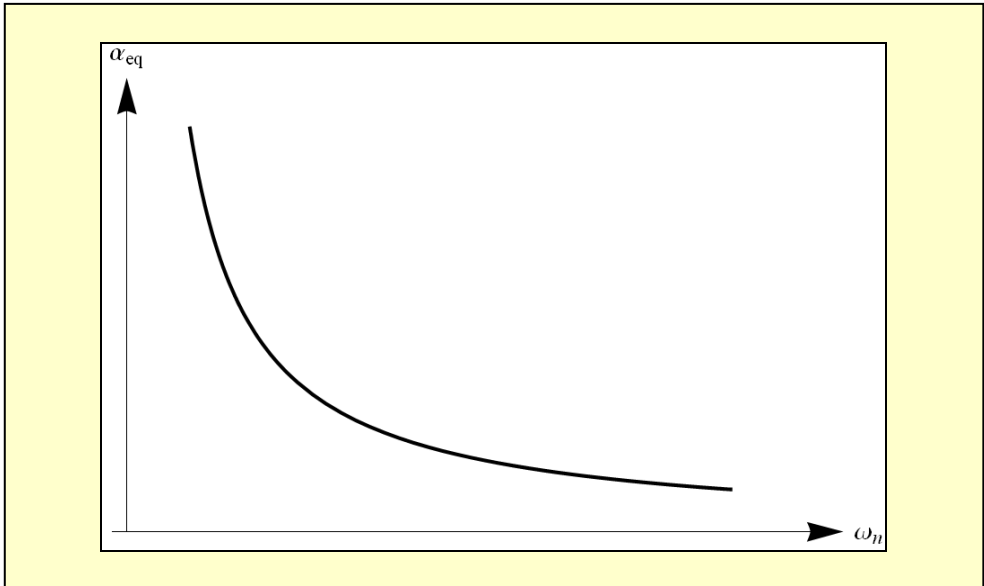

Fig. 2.27 Dependence of equivalent damping ratio on the natural frequency value

The transient force is now equal to the spring force

$$F_T = F_k = kq \qquad (2.130)$$

2.5.5.7. Rayleigh Damping

The Rayleigh model is a combination of the structural and mass models of damping. In this model it is assumed that the damping force is partly proportional to velocity of elastic reaction and partly proportional to the momentum of the mass. Thus, the damping coefficient is assumed to have the form

$$c = \kappa k + \mu m \qquad (2.131)$$

and the equivalent damping ratio is given by

$$\alpha_{eq} = \frac{\kappa \omega_n}{2} + \frac{\mu}{2\omega_n} \qquad (2.132)$$

The Rayleigh model takes into account both the viscous damping representing an environmental damping by external forces acting on the structure (external damping), and the damping due to the inherent properties of the material of the structure (internal damping).

The dependence of the equivalent damping ratio on the natural frequency value for the Rayleigh damping is shown in Fig. 2.28.
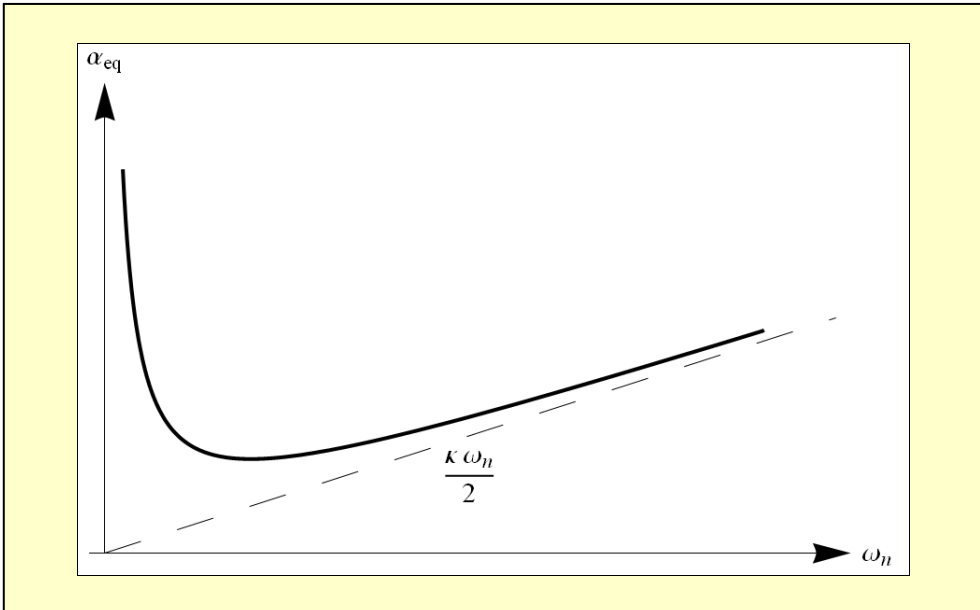

Fig. 2.28 Dependence of equivalent damping ratio from the natural frequency value

The transient force is now equal to

$$F_T = k(\kappa \dot{q} + q) \qquad (2.133)$$

CHAPTER 3

Multi-Degree-of-Freedom (MDOF) Systems

3. Multi-Degree-of-Freedom System (MDOF)

In reality, structures are not built of separate mass points, but consist of a continuous mass also called distributed mass. Such systems have an infinite number of degrees of freedom. However, it is virtually impossible to find dynamical solutions to any but the most simple of such systems. In general, it is necessary to discretize systems, i.e. replace infinite-number-of-degrees-of-freedom systems with simplified models – finite-number-of-degrees-of-freedom systems which are also called Multiple-Degree-of-Freedom (MDOF) systems.

A model which contains a finite number of degrees of freedom is called a discrete model. Discretization concerns the process of transferring continuous models and equations into discrete ones. Discretization can be realized as a mathematical approximation or as a granulation of masses. Discrete models can consist of clearly distinguishable, separate masses, called lumped masses.

Lumped mass models are created from continuous structures by replacing the distributed mass elements with a given number of lumped masses – the larger the number of masses, the better the approximation to the real structure. Furthermore, the masses (or lumped masses) may be interconnected by rigid elements; in such cases, the whole group acts as one rigid body possessed of both mass and moment of inertia. Mass points have translational degrees of freedom only, while the rigid bodies additionally have rotational degrees of freedom. The number of masses that may be used to represent a system is unlimited.

3.1. Degrees of Freedom (DOF)

Degrees of freedom are the ways in which the space configuration of a mechanical system may change, i.e. the independent movements the system can possibly undergo.

Degrees of freedom are also independent displacements and/or rotations that specify the orientation of the body or system.

3.2. Number of Degrees of Freedom

The number of degrees of freedom d of a mechanical system is equal to the minimum number of independent coordinates required to define completely the position of all parts of the system (configuration of a mechanical system) at any instant in time. In general, it is equal to the number of possible independent displacements. The numbers of degrees of freedom of a free (unconstrained) point and a free rigid body in space are shown in Fig. 3.1. The number of degrees of freedom of a free point and a free rigid body in a plane are shown in Fig. 3.2.

$$d = d_\Delta + d_\varphi \qquad (3.1)$$

where

d_Δ – number of translational degrees of freedom

d_φ – number of rotational degrees of freedom

3.3. Systems of Coordinates

In a dynamical analysis of MDOF systems three types of coordinates are used.

3.3.1. External Coordinates

The system of external coordinates is a fixed inertial set of reference axes (for instance the Cartesian coordinate system x_1, x_2, x_3, or x, y, z) useful for defining the configuration of a dynamic structure in a static equilibrium state.
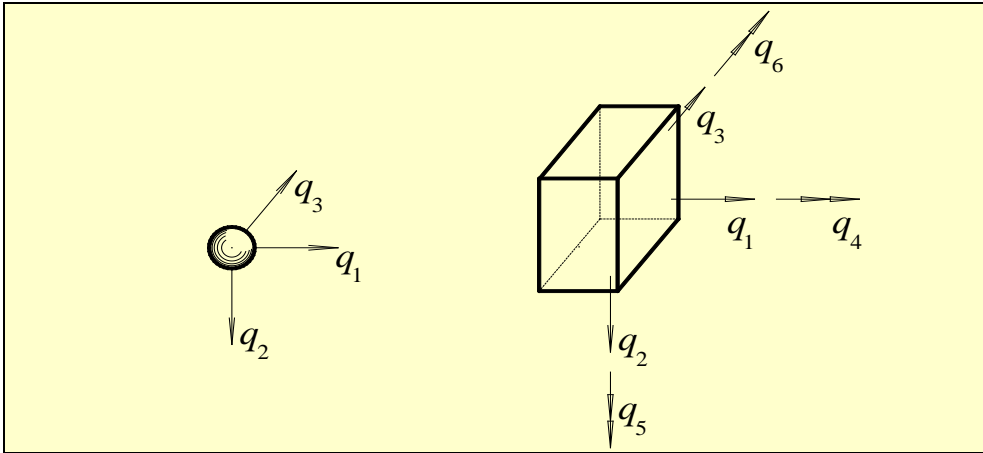
Fig. 3.1 Degrees of freedom for a free point and a free rigid body in space
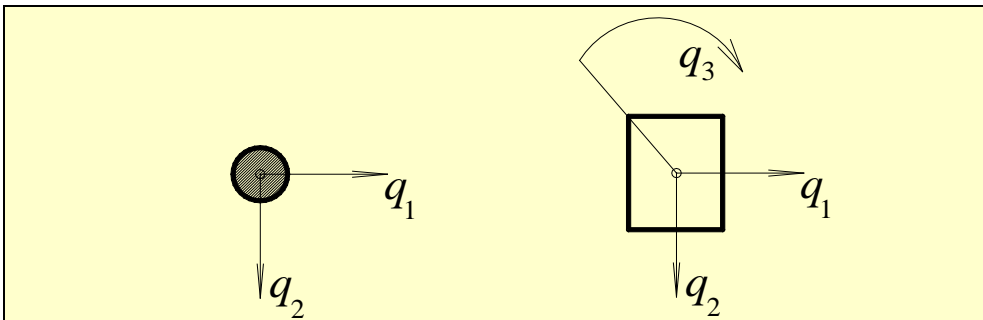
Fig. 3.2 Degrees of freedom for a free point and a free rigid body in a plane

3.3.2. Local Coordinates

Local coordinates u_i are dependent on time. They describe the movement of system elements from the static equilibrium state. Usually, but not necessarily, they describe the possible displacements of elements. Local coordinates are associated with: mass centers of masses and rigid bodies, points in which the springs and dampers are connected to the structure, points in which forces act on the structure and other points whose displacements are important for the dynamic description of the structure. Local

coordinates may be of a translational or a rotational type, see Fig. 3.3, Fig. 3.4, Fig. 4.4, Fig. 4.5, and Fig. 4.6.

3.3.3. Generalized Coordinates

Generalized coordinates (Lagrange's generalized coordinates vector **q**) are dependent on time. They are a set of coordinates used to describe the configuration of a system relative to some reference configuration. The expression "generalized" is a remnant of a time when Cartesian coordinates were the standard. Generalized coordinates may be of a translational or a rotational type, Fig. 3.3 – Fig. 3.6, Fig. 4.3, Fig. 4.4, and Fig. 4.5.

A restriction for choosing a set of generalized coordinates is that they have to unequivocally define any possible configuration of the system relative to the reference configuration. That is to say, with the use of those coordinates it must be possible to determine all local movements of all elements of the whole system. The generalized coordinates are chosen to be independent of one another. The number of independent generalized coordinates n is defined by the number of degrees of freedom of the system d. Usually, generalized coordinates are related to the mass centers' position or rigid bodies mass centers' positions, but as a rule, they are related to points of connections of masses or mass rigid bodies to the structure. In general, these points do not have to be mass centers. The number of generalized coordinates is then equal to the number of dynamic degrees of freedom (minimal base $n = d$). Nevertheless, there are some situations when it is more convenient to assume the number of generalized coordinates to be greater than the number of dynamic degrees of freedom $n > d$. These additional generalized coordinates are then usually related to the position of forces, springs or dampers, which are connected to the structure in points not related to the mass points, Fig. 3.3. There may also be other reasons to assume $n > d$.
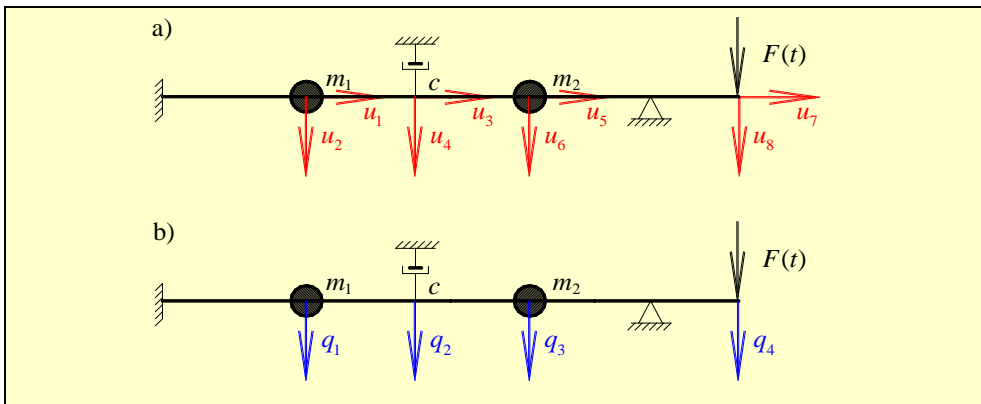
Fig. 3.3. The additional generalized coordinates q_2 and q_4 not related to the mass

Apart from practical reasons, all sets of generalized coordinates are equally good. The physics of the system are independent of the choice made between those sets. However, for practical reasons, some sets of coordinates are more useful than others – some are more optimally adapted to the system, and will make the solution of its equations of motion easier than others.

3.4. Defining a System and its Excitation

In more complex and complicated MDOF systems it is easier to derive equations of motion in terms of the energies of the system with the use of the Lagrangian equations. This approach is realized with accordance to a special procedure whose algorithm requires defining the system and its excitation in a specific way. This procedure is shown below.

3.4.1. Structures with Spring Elements – Stiffness Matrix

3.4.1.1. Stiffness Matrix in Local Coordinates Base of Spring Elements

Let us assume that there are some springs in the system, whose stiffnesses k_j are assembled in a diagonal matrix

$$\{\mathbf{k}\} = diag(k_1 \quad k_2 \quad k_3 \quad \ldots) = \begin{bmatrix} k_1 & 0 & 0 & \cdots \\ 0 & k_2 & 0 & \cdots \\ 0 & 0 & k_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{3.2}$$

The order of the elements in the diagonal matrix $\{\mathbf{k}\}$ corresponds to the order of local coordinates u_j in vector

$$\mathbf{u} = [u_1 \quad u_2 \quad u_3 \quad \ldots]^T \tag{3.3}$$

The elements of the vector $\mathbf{u}$ are the changes in length of springs.

3.4.1.2. Transformation from Generalized to Local Coordinates

The linear transformation of the generalized coordinates to the local coordinates vector is

$$\mathbf{u} = \mathbf{A}_k \mathbf{q} \tag{3.4}$$

where

$\mathbf{A}_k$ is the transformation matrix whose elements depend on the configuration of the structure only; the dimension of this transformation matrix is (number of local coordinates of spring elements) $\times$ (number of generalized coordinates)

3.4.1.3. Stiffness Matrix in Generalized Coordinates Base

The potential energy can be written in the form

$$E_p = \frac{1}{2} \sum_j k_j u_j^2 = \frac{1}{2} \mathbf{u}^T \{\mathbf{k}\} \mathbf{u} \tag{3.5}$$

The potential energy in the generalized base of coordinates, after substitution of Eq. (3.4) into formula Eq. (3.5), yields the expression

$$E_p = \frac{1}{2}\,\mathbf{q}^T \mathbf{A}_k^T \{\mathbf{k}\} \mathbf{A}_k \mathbf{q} = \frac{1}{2}\,\mathbf{q}^T \mathbf{K}\,\mathbf{q}$$

(3.6)

As it can be seen, the potential energy is given by the positive-definite quadratic form of the Lagrange's generalized coordinates $\mathbf{q}$, where $\mathbf{K}$ is the stiffness matrix and $\mathbf{q}^T$ is the transposed vector $\mathbf{q}$. Stiffness matrix $\mathbf{K}$ is non-singular, symmetric and positive-definite, thus $\det \mathbf{K} > 0$, and can be achieved from formula

$$\mathbf{K} = \mathbf{A}_k^T \cdot \{\mathbf{k}\} \cdot \mathbf{A}_k$$

(3.7)

The flexibility matrix is inverse to stiffness matrix

$$\mathbf{D} = \mathbf{K}^{-1}$$

(3.8)

Remark:

An ($n \times n$) real symmetric matrix $\mathbf{M}$ is positive-definite if $\mathbf{x}^T \mathbf{M} \mathbf{x} > 0$ for all non-zero vectors $\mathbf{x}$ with real entries ($\mathbf{x} \in \Re^n$), where $\mathbf{x}^T$ denotes the transpose of vector $\mathbf{x}$. All eigenvalues of positive-definite matrix $\mathbf{M}$ are positive. Positive-definite matrix $\mathbf{M}$ is always invertible (nonsingular) thus $\det \mathbf{M} > 0$.

3.4.2. Structures with Mass Elements – Mass (Inertia) Matrix

3.4.2.1. Transformation from Generalized to Local Coordinates

For small vibration, the local coordinates are the linear transformation of the generalized coordinate

$$\mathbf{u} = \mathbf{A}_m \mathbf{q}$$

(3.9)

where
$\mathbf{u}$ is the local coordinate of mass elements vector

$$\mathbf{u} = [u_1 \quad u_2 \quad u_3 \quad \ldots]^T$$

(3.10)

and

$\mathbf{A}_m$ is the transformation matrix whose elements depend on the configuration of the structure only; the dimension of this transformation matrix is (number of local coordinates of mass elements) $\times$ (number of generalized coordinates $n \geq d$)

3.4.2.2. Inertia Matrix in Local Coordinates Base of Mass Centers

If local coordinates can be assumed to be located in points which are the mass center of every mass or rigid body, then the inertia matrix has the diagonal form

$$\{\mathbf{m}\} = diag\,(m_1 \quad m_2 \quad m_3 \quad \ldots) = \begin{bmatrix} m_1 & 0 & 0 & \cdots \\ 0 & m_2 & 0 & \cdots \\ 0 & 0 & m_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{3.11}$$

where m_j are the masses *sensu stricto* or mass moments of inertia i.e. $m_j = J_j$, arranged with accordance to the elements of vector $\mathbf{u}$ respectively, Eq. (3.10).

3.4.2.3. Inertia Matrix in Generalized Coordinates Base

The kinetic energy of the whole system can be expressed by

$$E_k = \frac{1}{2}\sum_j m_j \dot{u}_j^2 = \frac{1}{2}\dot{\mathbf{u}}^T \{\mathbf{m}\}\dot{\mathbf{u}} \tag{3.12}$$

This formula is valid if the local coordinates are reduced to center points of masses and the translational coordinates system is a Cartesian one (in Polish, orthocartesian).

Substituting Eq. (3.9) into Eq. (3.12) it is possible to write

$$E_p = \frac{1}{2}\dot{\mathbf{q}}^T \mathbf{A}_m^T \{\mathbf{m}\} \mathbf{A}_m \dot{\mathbf{q}} = \frac{1}{2}\dot{\mathbf{q}}^T \mathbf{B} \dot{\mathbf{q}} \tag{3.13}$$

and finally

$$\mathbf{B} = \mathbf{A}_m^T \cdot \{\mathbf{m}\} \cdot \mathbf{A}_m \tag{3.14}$$

where $\mathbf{B}$ is the square and symmetric matrix of inertia in a generalized coordinate base. In a minimal base of generalized coordinates $n = d$, matrix $\mathbf{B}$ is non-singular and positive-definite, thus $\det \mathbf{B} > 0$. If $n > d$, inertia matrix $\mathbf{B}$ is non-negatively definite and singular ($\det \mathbf{B} = 0$).

Illustrative Example 3.1
A local and generalized coordinates in a plane system is shown in Fig. 3.4. This is the plane rigid body, which mass is m and polar mass moment of inertia about the axe through the mass center point J_O is given. The example shows haw to determine a mass matrix $\mathbf{B}$ in the generalized coordinate base.
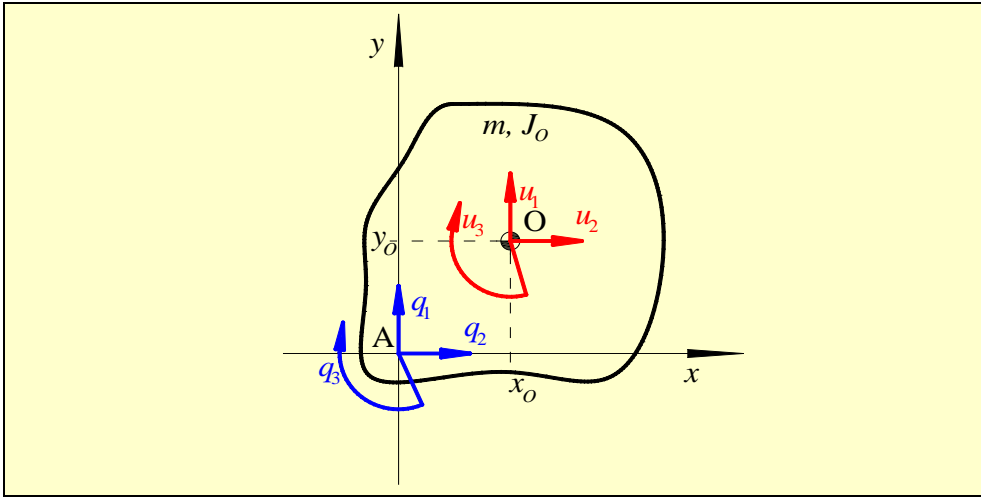
Fig. 3.4 Local and generalized coordinates in a plane system (plane rigid body: mass m and polar mass moment of inertia about the axe through the mass center point)

$$\mathbf{u} = [u_1 \quad u_2 \quad u_3]^T \tag{3.15}$$

$$\mathbf{q} = [q_1 \quad q_2 \quad q_3]^T \tag{3.16}$$

$$\mathbf{A}_m = \begin{bmatrix} 1 & 0 & -x_O \\ 0 & 1 & y_O \\ 0 & 0 & 1 \end{bmatrix} \tag{3.17}$$

$$\{\mathbf{m}\} = diag\,(m \quad m \quad J_O) = \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & J_O \end{bmatrix} \tag{3.18}$$

$$\mathbf{B} = \mathbf{A}_m^T \cdot \{\mathbf{m}\} \cdot \mathbf{A}_m = \begin{bmatrix} m & 0 & -mx_O \\ 0 & m & my_O \\ -mx_O & my_O & m(x_O^2 + y_O^2) + J_O \end{bmatrix} = \begin{bmatrix} m & 0 & -S_y \\ 0 & m & S_x \\ -S_y & S_x & J_A \end{bmatrix} \tag{3.19}$$

where

$S_x = my_O$ static moment of mass about the axis x

$S_y = mx_O$ static moment of mass about the axis y

$J_A = m(x_O^2 + y_O^2) + J_O$ polar mass moment of inertia about the axis through the dynamic center – point A

3.4.3. Structures with Damping Elements – Damping matrix

Rayleigh Dissipation Function describes the power of resistance forces, i.e. the work of these forces in time, which occurs in systems performing small oscillations. These forces are assumed to be proportional to velocities. The Rayleigh dissipation function, also known as the dissipation function, is given by the positive-definite quadratic form of the generalized velocities $\dot{\mathbf{q}}$

$$\Phi = \frac{1}{2}\dot{\mathbf{q}}^{\mathrm{T}}\mathbf{C}\dot{\mathbf{q}} \tag{3.20}$$

where $\mathbf{C}$ is the square and symmetric damping matrix in the generalized coordinate base.

In a general case, two kinds of damping can exist in civil engineering structures: apparent (dampers) and non-apparent (structural damping). With accordance to this classification, a damping matrix can be formulated as a sum of two damping matrices

$$\mathbf{C} = \mathbf{C}_1 + \mathbf{C}_2 \tag{3.21}$$

where

$\mathbf{C}_1$ damping matrix achieved with the use of hypothesis of damping, which represents the structural damping and the environment influence (Eq. 3.22)

$\mathbf{C}_2$ damping matrix which represents the influence of the dampers (chapters 3.4.3.1 to 3.4.3.3)

Usually, the damping matrix, by analogy to Eq. (2.128), is assumed to be proportional to the mass matrix (mass damping) or, by analogy to Eq. (2.120), is proportional to the stiffness matrix (structural damping)

$$\mathbf{C}_1 = \mu\mathbf{B} \qquad\qquad \mathbf{C}_1 = \kappa\mathbf{K} \tag{3.22}$$

It is best to assume that, by analogy to Eq. (2.132), the damping matrix is proportional to both these matrices (Rayleigh damping)

$$\mathbf{C}_1 = \kappa\mathbf{K} + \mu\mathbf{B} \tag{3.23}$$

where finally $\mathbf{C}$ is the square and symmetric damping matrix in the generalized coordinate base.

3.4.3.1. Transformation from Generalized to Local Coordinates

For small vibration, local coordinates are the linear transformation of the generalized coordinates

$$\mathbf{u} = \mathbf{A}_d\mathbf{q} \tag{3.24}$$

where

$\mathbf{u}$ is the local coordinate of mass elements vector

$$\mathbf{u} = [u_1 \quad u_2 \quad u_3 \quad \ldots]^T \tag{3.25}$$

and

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$\mathbf{A}_d$ is the transformation matrix whose elements depend on the configuration of the structure only; the dimension of this transformation matrix is (number of local coordinates of damped elements) × (number of generalized coordinates $n \geq d$)

3.4.3.2. Damping Matrix in Local Coordinates Base of Damped Elements

If local coordinates can be assumed to be located along damping elements, then the damping matrix has the diagonal form

$$\{\mathbf{c}\} = diag(c_1 \quad c_2 \quad c_3 \quad \ldots) = \begin{bmatrix} c_1 & 0 & 0 & \cdots \\ 0 & c_2 & 0 & \cdots \\ 0 & 0 & c_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{3.26}$$

3.4.3.3. Damping Matrix in Generalized Coordinates Base

The Rayleigh dissipation function of the whole system can be expressed by

$$\Phi = \frac{1}{2}\sum_j c_j \dot{u}_j^2 = \frac{1}{2}\dot{\mathbf{u}}^T\{\mathbf{c}\}\dot{\mathbf{u}} \tag{3.27}$$

where c_j are the damping coefficients of dampers arranged with accordance to the elements of vector $\mathbf{u}$ respectively, Eq. (3.26).

Substituting Eq. (3.24) into Eq. (3.27) it is possible to write

$$\Phi = \frac{1}{2}\dot{\mathbf{q}}^T\mathbf{A}_d^T\{\mathbf{c}\}\mathbf{A}_d\dot{\mathbf{q}} = \frac{1}{2}\dot{\mathbf{q}}^T\mathbf{C}\dot{\mathbf{q}} \tag{3.28}$$

and finally

$$\mathbf{C}_2 = \mathbf{A}_d^T \cdot \{\mathbf{c}\} \cdot \mathbf{A}_d \tag{3.29}$$

where $\mathbf{C}$ is the square and symmetric damping matrix in the generalized coordinate base.

3.4.4. Excitation Forces

Work W is the transfer of energy that occurs when a force acts on a body, and is calculated as a dot product of the vector of force $\mathbf{F}$ and displacement $\mathbf{q}$ (displacement of the point the force is acting on). If a body is moving in such a way that the force has a component in a direction perpendicular to the direction of the body's motion, the work of that component is equal to zero. In any situation, the work is given by the linear form of the coordinates $\mathbf{q}$

$$W = \mathbf{F} \cdot \mathbf{q} \tag{3.30}$$

3.4.4.1. Transformation from Generalized to Local Coordinates

If local coordinates can be assumed to be located in the points the excitation forces act on, these local coordinates are, for small vibration, the linear transformation of the generalized coordinate

$$\mathbf{u} = \mathbf{A}_f\,\mathbf{q} \tag{3.31}$$

where

$$\mathbf{u} = [u_1 \quad u_2 \quad u_3 \quad \ldots]^T \tag{3.32}$$

and

$\mathbf{u}$ is the local coordinates of forces vector

$\mathbf{A}_f$ is the transformation matrix whose elements depend on the configuration of the structure only; the dimension of this transformation matrix is (number of local coordinates of forces) × (number of generalized coordinates $n \geq d$)

3.4.4.2. Excitation Force Vector in Local Coordinates Base

The excitation forces vector, in local base of coordinates Eq. (3.32), has the form

$$\mathbf{P} = [P_1 \quad P_2 \quad P_3 \quad \ldots]^T \tag{3.33}$$

3.4.4.3. Excitation Force Vector in Generalized Coordinates Base

The virtual work of excitation forces on the displacements described by Eq. (3.32) can be written as follows

$$W = \sum_j P_j u_j = \mathbf{u}^T \mathbf{P} = \mathbf{q}^T \mathbf{A}_f^T \mathbf{P} = \mathbf{q}^T \mathbf{F}. \tag{3.34}$$

where

$$\mathbf{F} = \mathbf{A}_f^T \mathbf{P} \tag{3.35}$$

The work of the external forces $\mathbf{F}$ is the linear form of the generalized coordinates $\mathbf{q}$.

3.5. Lagrangian Equations

As has been stated in Chapter 2.2, formulating a differential equation of motion for a vibrating system by applying d'Alambert's Principle may be complicated even for SDOF systems. Therefore, in MDOF systems, which are usually more complex, it is easier to derive these equations of motion in terms of the energies of the system with the use of the Lagrangian equations

$$\frac{d}{dt}\frac{\partial E_k}{\partial \dot{q}_i} - \frac{\partial E_k}{\partial q_i} + \frac{\partial \Phi}{\partial \dot{q}_i} + \frac{\partial E_p}{\partial q_i} = \frac{\partial W}{\partial q_i}, \quad i = 1,2,\ldots,n \tag{3.36}$$

or in vector notation

$$\frac{d}{dt}\,grad\,E_k(\dot{\mathbf{q}}) - grad\,E_k(\mathbf{q}) + grad\,\Phi(\dot{\mathbf{q}}) + grad\,E_p(\mathbf{q}) = grad\,W(\mathbf{q}) \tag{3.37}$$

where

E_k total kinetic energy of the system

E_p total potential energy of the system

Φ Rayleigh Dissipation Function

W virtual work of external generalized forces (the portion of the forces not related to the potential energy of the system (gravity and spring forces appear in the potential energy expressions and are not included here)

q_i generalized coordinate (displacement)

$\mathbf{q}$ generalized coordinates vector

$\dot{q}_i$ velocity at generalized coordinate q_i

$\dot{\mathbf{q}}$ generalized velocities vector

In the case of small vibration around the equilibrium point the kinetic energy in general does not depend on generalized displacements q_i . Then

$$\frac{\partial E_k}{\partial q_i} = 0 \qquad\qquad grad\,E_k(\mathbf{q}) = \mathbf{0} \tag{3.38}$$

and the Lagrangian equations take the form

$$\frac{d}{dt}\frac{\partial E_k}{\partial \dot{q}_i} + \frac{\partial \Phi}{\partial \dot{q}_i} + \frac{\partial E_p}{\partial q_i} = \frac{\partial W}{\partial q_i}, \quad i = 1,2,\ldots,n \tag{3.39}$$

or in vector notation

$$\frac{d}{dt}\,grad\,E_k(\dot{\mathbf{q}}) + grad\,\quad(\dot{\mathbf{q}}) + grad\,E_p(\mathbf{q}) = grad\,W(\mathbf{q}) \tag{3.40}$$

3.6. Equation of Motion

After substitution of the formulas Eqs. (3.6), (3.13), (3.20) and (3.34) into Eq. (3.40), and after differentiation of these expressions with respect to each chosen coordinate, one can receive an n number of equations of motion in the generalized coordinate base. In the matrix form, this system may be written as

$$\mathbf{B}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{F}(t) \tag{3.41}$$

where

    B   mass (inertia) matrix of a system
    C   damping matrix of a system
    K   stiffness matrix of a system
    $\mathbf{F}(t)$ vector of external generalized forces acting on a system

    $\mathbf{q}, \dot{\mathbf{q}}, \ddot{\mathbf{q}}$ generalized coordinates, velocities and accelerations vectors respectively

The above is the equation of motion formulated with the use of the Displacement Method to calculate the stiffness matrix. If the equation of motion were to be formulated with the use of the Force Method to calculate the flexibility matrix, it would have the form

$$\mathbf{D}\mathbf{B}\ddot{\mathbf{q}} + \mathbf{D}\mathbf{C}\dot{\mathbf{q}} + \mathbf{q} = \mathbf{D}\mathbf{F}(t) \tag{3.42}$$

If both methods (Displacement and Force Methods) make use of the same generalized coordinate base $\mathbf{q}$, it is possible to transform one equation into the other by a simple matrix operation without formally using either of the methods.

For instance, in order to obtain the Eq. (3.42) from the Eq. (3.41), it is enough to premultiply the Eq. (3.41) by the flexibility matrix calculated as $\mathbf{D} = \mathbf{K}^{-1}$; and *vice versa*, Eq. (3.42) should be premultiplied by the stiffness matrix calculated from the flexibility matrix according the to the formula $\mathbf{K} = \mathbf{D}^{-1}$ in order to obtain the Eq. (3.41).

3.7. Systems with Elastic and Inertial Coupling

Static (Elastic) coupling

If there are off-diagonal terms in the stiffness matrix $\mathbf{K}$ in the matrix equation of motion Eq. (3.41), this indicates that elastic coupling exists.

Inertial coupling

If there are off-diagonal terms in the mass matrix $\mathbf{B}$ in the matrix equation of motion Eq. (3.41), this indicates that inertial coupling exists.

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3.7.1. System with Static Coupling

Let us consider the two-degree-of-freedom system shown in Fig. 3.5
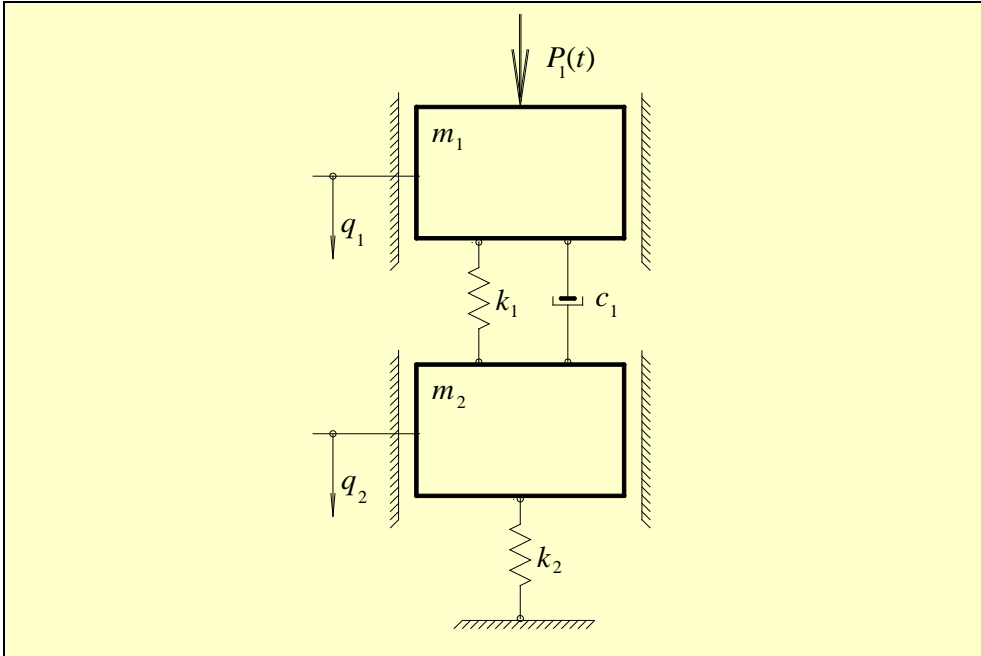
Fig. 3.5 Two-degree-of-freedom system – generalized coordinate vector **q**

The system has two degrees of freedom. The generalized coordinate vector is

$$\mathbf{q} = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} \tag{3.43}$$

The equation of motion can be written in the matrix form

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{bmatrix} + \begin{bmatrix} c_1 & -c_1 \\ -c_1 & c_1 \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} + \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1+k_2 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} P_1 \\ 0 \end{bmatrix} \tag{3.44}$$

It is apparent that the equations in Eq. (3.44) are coupled, as off-diagonal terms occur in the stiffness matrix. As these terms do not occur in the mass matrix, only elastic coupling exists.

3.7.2. System with Inertial Coupling

Let us consider the two-degree-of-freedom system shown in Fig. 3.6
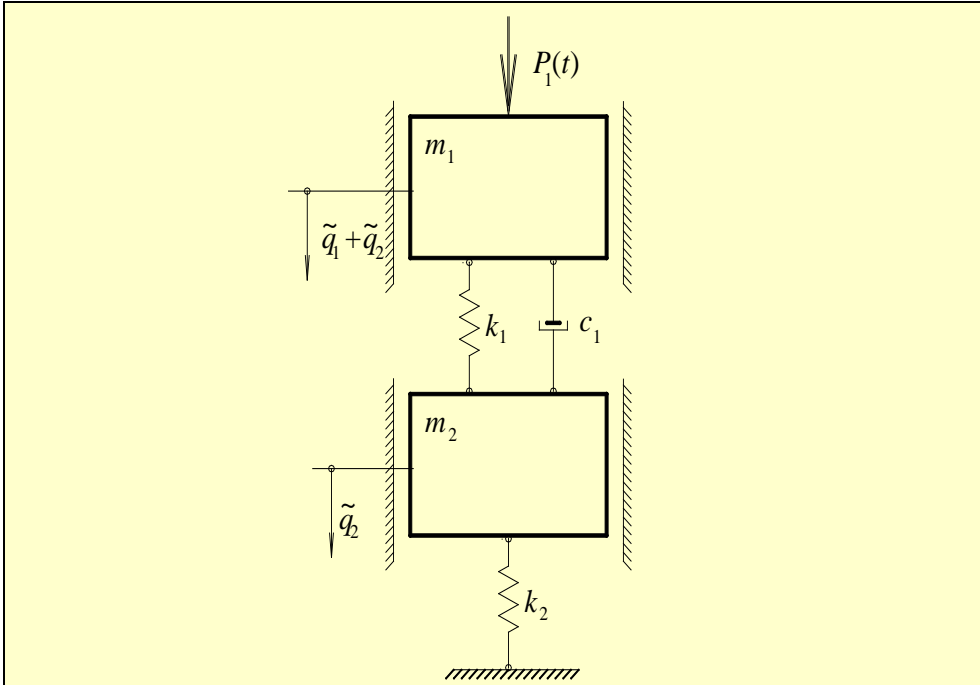
Fig. 3.6 Two-degree-of-freedom system – generalized coordinate vector $\tilde{\mathbf{q}}$

This is the same two-degrees-of-freedom system as in Fig. 3.5, but a different generalized coordinate system is chosen. The generalized coordinate vector is now

$$\tilde{\mathbf{q}} = \begin{bmatrix} \tilde{q}_1 \\ \tilde{q}_2 \end{bmatrix} = \begin{bmatrix} q_1 - q_2 \\ q_2 \end{bmatrix} \tag{3.45}$$

Now the matrix equation of motion has the form

$$\begin{bmatrix} m_1 & m_1 \\ m_1 & m_1 + m_2 \end{bmatrix} \begin{bmatrix} \ddot{\tilde{q}}_1 \\ \ddot{\tilde{q}}_2 \end{bmatrix} + \begin{bmatrix} c_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\tilde{q}}_1 \\ \dot{\tilde{q}}_2 \end{bmatrix} + \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix} \begin{bmatrix} \tilde{q}_1 \\ \tilde{q}_2 \end{bmatrix} = \begin{bmatrix} P_1 \\ P_1 \end{bmatrix} \tag{3.46}$$

It is apparent that the equations in Eq. (3.46) are also coupled, as off-diagonal terms occur in the mass matrix. As these terms do not occur in the stiffness matrix, only inertial coupling exists.

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3.7.3. System with Simultaneous Static and Inertial Coupling

As the type of coupling depends on the choice of the generalized coordinates system, it is possible to choose such generalized coordinates that both static and inertial coupling will occur simultaneously. It seems also to be possible to find such a generalized coordinate system for which the equations of motion will be uncoupled.

3.7.4. System without Coupling (Decoupled System)

The generalized coordinates system for which there is no coupling at all is called the principal generalized coordinates system. This case will be further investigated in Chapters 3.12 and 3.13.

Conclusions:

- the coupling of the equations of motion in MDOF systems is not a distinctive feature of the system but depends on the choice of the generalized coordinate system

- the MDOF system equations of motion can be coupled in three ways: inertially, elastically or inertially and elastically simultaneously

- uncoupled systems of equations, in which no coupling exists at all, are also possible

---

3.8. Natural Frequencies and Normal Modes of Vibration

"The number of natural frequencies of vibration of any system is equal to the number of degrees-of-freedom; thus the system having distributed parameters has an infinite number of natural frequencies. At a given time, such a system usually vibrates with appreciable amplitude at only a limited number of frequencies, often at only one. With each natural frequency is associated a shape, called the normal or natural mode, which is assumed by the system during free vibration at the frequency. For example, when a uniform beam with simple supported or hinged ends vibrates laterally at its lowest or fundamental natural frequency, it assumes the shape of the half sine wave; this is a normal mode of vibration. When vibrating in this manner, the beam behaves as a system with a single degree-of-freedom, since its configuration at any time can be defined by giving the deflection of the center of the beam. When any linear system, i.e. one in which the elastic restoring force is proportional to the deflection, executes free vibration in a single natural mode, each element of the system except those at the supports and nodes executes simple harmonic mode about its equilibrium position. All possible free vibration of any linear system is made up of superposed vibrations in the normal modes at the corresponding natural frequencies. The total motion at any point of the system is the sum of the motions resulting from the vibration in the respective modes.

There are always nodal points, lines, or surfaces, i.e. points which do not move, in each of the normal modes of vibration of any system. For the fundamental mode, which corresponds to the lowest natural frequency, the supported or fixed points of the

system usually are the nodal points; for other modes, there are additional nodes. In the modes of vibration corresponding to the patterns. In certain problems are, since a particular mode usually will not be excited by a force acting at a nodal points.", [1].

3.8.1. Eigenproblem

The problem of free vibrations requires that the force vector be equal to zero in Eq. (3.41). If the system is also undamped, the equation of motion can be written in form

$$\mathbf{B\ddot{q}} + \mathbf{Kq} = \mathbf{0} \tag{3.47}$$

where

        $\mathbf{B}$ mass matrix of a system

        $\mathbf{K}$ stiffness matrix of a system

        $\mathbf{q}$, $\ddot{\mathbf{q}}$ generalized coordinates and accelerations vectors

For the free vibrations of the undamped structure, one can guess (Lucky Guess Method) the form of the solutions of Eq. (3.47)

$$\mathbf{q}(t) = \mathbf{a}\sin(\omega t + \varphi) \tag{3.48}$$

Two times differentiation of expression Eq. (3.48) with respect to time leads to formula

$$\ddot{\mathbf{q}} = -\omega^2 \mathbf{q} \tag{3.49}$$

The substitution of Eq. (3.49) into Eq. (3.47) gives

$$(\mathbf{K} - \omega^2 \mathbf{B})\mathbf{q} = \mathbf{0} \tag{3.50}$$

which, in this case, is a set of n homogeneous algebraic linear equations with n unknown displacements q_j ($j = 1,\ldots,n$) and an unknown parameter ω^2. The formulation of Eq. (3.50) is an important mathematical problem known as the eigenproblem. Its nontrivial solution, that is, the solution for which not all $q_j = 0$, requires that the determinant of the matrix factor of $\mathbf{q}$ be equal to zero, namely

$$\det(\mathbf{K} - \omega^2 \mathbf{B}) = 0 \qquad \text{or} \qquad \det(\mathbf{K} - \lambda \mathbf{B}) = 0 \tag{3.51}$$

In general, the expansion of the determinant in Eq. (3.51) results in a polynomial equation of degree n in $\lambda = \omega^2$ which should be satisfied for n eigenvalues of $\lambda_i = \omega_{ni}^2$ – real roots of polynomial Eq. (3.51). This polynomial is known as the characteristic equation of the system.

For any one natural frequency ω_{ni} (eigenvalue of $\lambda_i = \omega_{ni}^2$) such a solution exists

$$\mathbf{q} = \mathbf{w}_i \tag{3.52}$$

that

$$(\mathbf{K} - \omega_{ni}^2\mathbf{B})\mathbf{w}_i = \mathbf{0} \tag{3.53}$$

The vector $\mathbf{w}_i$ is called an eigenvector (*i*-th normal or natural mode of vibration or modal shape). The eigenvector coordinates are generalized displacements, which describe the modal shape, that is they specify how, for each natural angular frequency ω_{ni}, the various elements of the system move in relation to each other.

It is easy to prove that each non-zero column of the adjugate (adjoint) matrix

$$\mathrm{adj}\mathbf{A}_i = \mathrm{adj}(\mathbf{K} - \omega_i^2\mathbf{B}) \tag{3.54}$$

is an eigenvector of Eq. (3.50).

Proof:
Let us consider the formula for the inverse matrix

$$\mathbf{A}_i^{-1} = \frac{\mathrm{adj}\mathbf{A}_i}{\det\mathbf{A}_i} \tag{3.55}$$

This formula can be written in another form, i.e.

$$\mathbf{A}_i^{-1}\det\mathbf{A}_i = \mathrm{adj}\mathbf{A}_i \tag{3.56}$$

Premultiplication of Eq. (3.56) by matrix $\mathbf{A}_i$ and postmultiplication by any non-zero vector $\mathbf{v}$ results in

$$\det\mathbf{A}_i \cdot \mathbf{v} = \mathbf{A}_i\,\mathrm{adj}\mathbf{A}_i \cdot \mathbf{v} \tag{3.57}$$

Since $\det\mathbf{A}_i = 0$ and designates a new vector $\mathbf{b} = \mathrm{adj}\mathbf{A}_i \cdot \mathbf{v}$, Eq. (3.57) can be written in the form

$$\det\mathbf{A}_i\,\mathbf{v} = \mathbf{A}_i\,(\mathrm{adj}\mathbf{A}_i \cdot \mathbf{v}) = \mathbf{A}_i\mathbf{b} = (\mathbf{K} - \omega_{ni}^2\mathbf{B})\mathbf{b} = \mathbf{0} \tag{3.58}$$

As $\mathbf{v}$ could be any vector it could also be a versor, with 1 on *j*-th position. Then the multiplication $\mathbf{A}_i\mathbf{v}$ takes out *j*-th column from the adjoint matrix. Comparing Eqs. (3.58) and (3.50), one can write down

$$\mathbf{w}_i = \mathbf{b} = \mathrm{adj}(\mathbf{K} - \omega_{ni}^2\mathbf{B}) \tag{3.59}$$

The eigenvectors can be normalized

$$\mathbf{w}_{i,\mathrm{norm}} = \mathbf{w}_i / N_i \tag{3.60}$$

usually using the norms

$$N_i = \|\mathbf{w}_i\| = \max_j |w_{ji}|$$ (3.61)

or

$$N_i = \|\mathbf{w}_i\| = \sqrt{\mathbf{w}_i^T \mathbf{B} \, \mathbf{w}_i}$$ (3.62)

3.8.2. Modal and Spectral Matrices

For each value of eigenfrequency ω_{ni} (natural angular frequency) satisfying the characteristic Eq. (3.51) one may solve Eq. (3.53). This solution, with an accuracy up to a constant multiplier (multiplicative constant), is the eigenvector

$$\mathbf{w}_i = \begin{bmatrix} q_{i1} \\ q_{i2} \\ \vdots \\ q_{in} \end{bmatrix} = \begin{bmatrix} w_{i1} \\ w_{i2} \\ \vdots \\ w_{in} \end{bmatrix}$$ (3.63)

The solutions Eq. (3.63) of Eq. (3.53) for $i=1,\ldots,n$ describe the normal modes (shapes) which may be conveniently arranged in the columns of a matrix known as the modal matrix $\mathbf{W}$, that is

$$\mathbf{W} = [\mathbf{w}_1 \quad \mathbf{w}_2 \quad \ldots \quad \mathbf{w}_n] = \begin{bmatrix} w_{11} & w_{12} & \ldots & w_{1n} \\ w_{21} & w_{22} & \ldots & w_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ w_{n1} & w_{n2} & \ldots & w_{nn} \end{bmatrix}$$ (3.64)

The n eigenvalues $\lambda_i = \omega_{ni}^2$ can be assembled into a diagonal matrix $\mathbf{\Omega}^2 = \{\mathbf{\omega}^2\}$ which is known as a spectral matrix of the eigenproblem Eq. (3.50), that is

$$\mathbf{\Omega}^2 = \{\mathbf{\omega}^2\} = \begin{bmatrix} \omega_{n1}^2 & 0 & \ldots & 0 \\ 0 & \omega_{n2}^2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \omega_{nn}^2 \end{bmatrix}$$ (3.65)

By using the modal and spectral matrices it is possible to assemble all of these relations into a single matrix equation

$$\mathbf{KW} = \mathbf{BW\Omega}^2$$ (3.66)

3.9. Normal and Natural Mode of Vibration

When a system is capable of vibrating with more than one frequency, but is actually vibrating freely at only one of its possible natural frequencies, the system is said to be vibrating in one of its "normal modes."

"A normal mode of vibration is a mode of vibration that is uncoupled from (i.e., can exist independently of) other modes of vibration of a system. When vibration of the system is defined as an eigenvalue problem, the normal modes are the eigenvectors and the normal mode frequencies are the eigenvalues. The term classical normal mode is sometimes applied to the normal modes of a vibrating systems characterized by vibration of each element of the system at the same frequency and phase. In general, classical normal modes exist only in systems having no damping or having particular type of damping.", [1].

"The natural mode of vibration is a mode of vibration assumed by a system when vibrating freely.", [1].

The mode of vibration associated with the lowest natural frequency of a system is referred to as the first (basic) mode. The next higher frequency is the second, and so on.

3.10. Orthogonality of Normal Modes (Orthogonality Principle)

The most important property of the normal modes is their orthogonality. For this reason the normal modes can be used to uncouple the matrix equations of motion. The solution of a set of separate differential equations is significantly easier than the solution of a set of coupled differential equations.

Premultiplication of Eq. (3.66) by matrix $\mathbf{W}^T$ yields equation

$$\mathbf{W}^T \mathbf{K} \mathbf{W} = \mathbf{W}^T \mathbf{B} \mathbf{W} \mathbf{\Omega}^2 \qquad (3.67)$$

After transposition of Eq. (3.67), and taking into account the symmetry of matrices $\mathbf{B}$ and $\mathbf{K}$, Eq. (3.67) can be written in form

$$\mathbf{W}^T \mathbf{K} \mathbf{W} = \mathbf{\Omega}^2 \mathbf{W}^T \mathbf{B} \mathbf{W} \qquad (3.68)$$

The left sides of Eqs. (3.67) and (3.68) are the same, thus the right sides must also be the same, that is

$$\mathbf{W}^T \mathbf{B} \mathbf{W} \mathbf{\Omega}^2 = \mathbf{\Omega}^2 \mathbf{W}^T \mathbf{B} \mathbf{W} \qquad (3.69)$$

In general, Eq. (3.69) is true only if matrix $\mathbf{W}^T \mathbf{B} \mathbf{W}$ is a diagonal matrix. That matrix is called the principal masses matrix and its elements – modal masses

$$\mathbf{W}^T \mathbf{B} \mathbf{W} = \{\mathbf{m_o}\} \qquad (3.70)$$

Substituting Eq. (3.70) into Eq. (3.68) implies that matrix $\mathbf{W}^T \mathbf{K} \mathbf{W}$ must also be a diagonal matrix (principal stiffnesses matrix) and its elements – modal stiffnesses

$$\mathbf{W}^T\mathbf{K}\mathbf{W} = \{\mathbf{k_o}\} = \{\mathbf{m_o}\}\{\omega^2\} \tag{3.71}$$

Formulas Eqs. (3.70) and (3.71) are valid if the normalization of modal vectors is done by using the norm of vector Eq. (3.65). The normalization of the modal vectors with the use of Eq. (3.66) yields formulas

$$\mathbf{W}^T\mathbf{B}\mathbf{W} = \{\mathbf{m_o}\} = \mathbf{I} \tag{3.72}$$

and

$$\mathbf{W}^T\mathbf{K}\mathbf{W} = \{\mathbf{k_o}\} = \left\{\omega^2\right\} \tag{3.73}$$

where $\mathbf{I}$ is an identity matrix.

If the flexibility matrix is used to formulate the equation of motion, Eq. (3.42), the reduced equation of motion has the form

$$\mathbf{D}\mathbf{B}\ddot{\mathbf{q}} + \mathbf{q} = \mathbf{0} \tag{3.74}$$

where $\mathbf{D}$ is a flexibility matrix. Now, Eq. (3.50) should be substituted with

$$(\mathbf{D}\mathbf{B} - \omega^{-2}\mathbf{I})\mathbf{q} = \mathbf{0} \tag{3.75}$$

and consequently, instead of Eq. (3.51)

$$\det(\mathbf{D}\mathbf{B} - \omega^{-2}\mathbf{I}) = 0 \tag{3.76}$$

instead of Eq. (3.52)

$$(\mathbf{D}\mathbf{B} - \omega^{-2}\mathbf{I})\mathbf{w}_i = \mathbf{0} \tag{3.77}$$

and etc.

Conclusions:

- It must be noticed that eigenvectors are orthogonal with respect to both the mass and the stiffness matrix, but eigenvectors **are not orthogonal** with respect to the flexibility matrix.

- Each eigenvector is determined in terms of an arbitrary constant and can normalized arbitrarily.

- If $n > d$ (base of generalized coordinates is not minimal) the eigenfrequency $\omega = \infty$ can appear. These solutions must be neglected.

3.11. Natural Vibration

3.11.1. Ambiguity of Term "Natural Vibration"

The term "natural vibration" is somewhat problematic due to the conventions of Polish terminology. Following Langer [5], the Polish term "*drgania własne*" (the lexical equivalent of the English term "natural vibration") does not relate to a physical phenomenon. It does not designate vibration, but a mathematical form of the general solution (total integral) of an inhomogeneous differential equation of motion without damping, Eq. (3.47), which describes a predisposition of the structure to vibrate freely with accordance to natural frequencies and natural (normal) forms of vibration.

In English terminology, the term "natural vibration" appears predominantly in connection to such terms as "frequency of natural vibration" or "mode of natural vibration". This indicates that "natural vibration" are identified with "free vibration". As Harris writes, "[t]he natural mode of vibration is a mode of vibration assumed by a system when vibrating freely.", [1]. Thus, the expression, as used in the English terminology, clearly describes a physical phenomenon.

Therefore, the term "natural vibration" may be ambiguous, as it will have one meaning when used in the sense attached to it in English terminology, and a different one if used as a translation of the Polish "*drgania własne*". It is suggested that the term "natural vibration" should be used uniformly in the meaning equivalent to free vibration to avoid this ambiguity.

3.11.2. Natural Vibration According to Langer's Meaning

As always for the second order differential linear equations, this solution depends on independent constants whose number is equal to the doubled number of the equations. This general solution can be expressed in form

$$\mathbf{q}(t) = \sum_{i=1}^{d} \mathbf{w}_i (s_i \sin \omega_{ni} t + c_i \cos \omega_{ni} t) = \mathbf{W} \cdot \{\sin\boldsymbol{\omega}_\mathbf{n}\mathbf{t}\} \cdot \mathbf{s} + \mathbf{W} \cdot \{\cos\boldsymbol{\omega}_\mathbf{n}\mathbf{t}\} \cdot \mathbf{c} \qquad (3.78)$$

where

$$\{\sin\boldsymbol{\omega}_\mathbf{n}\mathbf{t}\} = \mathrm{diag}(\sin \omega_{n1} t \quad \sin \omega_{n2} t \quad \dots \quad \sin \omega_{nd} t)$$

$$\{\cos\boldsymbol{\omega}_\mathbf{n}\mathbf{t}\} = \mathrm{diag}(\cos \omega_{n1} t \quad \cos \omega_{n2} t \quad \dots \quad \cos \omega_{nd} t) \qquad (3.79)$$

$$\mathbf{s} = [s_1 \quad s_2 \quad \dots \quad s_d]^T \quad \mathbf{c} = [c_1 \quad c_2 \quad \dots \quad c_d]^T$$

Elements of the vectors $\mathbf{s}$ and $\mathbf{c}$ are constants of integration. As it can be seen, the time history of solution of Eq.(3.78) is a combination of simple harmonic oscillations with natural frequencies ω_{ni}, whose amplitudes are proportional to eigenvectors $\mathbf{w}_i$.

3.12. Free Vibration

3.12.1. Undamped Free Vibration

Free vibration (natural vibration) is a physical phenomenon which occurs in an undamped MDOF system when there is no excitation and the initial conditions are given: $\mathbf{q}(0) = \mathbf{q}_o$ and $\dot{\mathbf{q}}(0) = \dot{\mathbf{q}}_o$. If the base of generalized coordinates is minimal ($n = d$) the solution Eq. (3.78) may be used to determinate the free vibration of the system, so

$$\begin{aligned} \mathbf{q}_o &= \mathbf{Wc} \quad \rightarrow \quad \mathbf{c} = \mathbf{W}^{-1}\mathbf{q}_o \\ \dot{\mathbf{q}}_o &= \mathbf{W} \cdot \{\boldsymbol{\omega}\} \cdot \mathbf{s} \quad \rightarrow \quad \mathbf{s} = \{\boldsymbol{\omega}^{-1}\} \cdot \mathbf{W}^{-1}\dot{\mathbf{q}}_o \end{aligned} \tag{3.80}$$

and the solution of Eq. (3.78) is

$$\mathbf{q}(t) = \mathbf{W} \cdot \{\cos\boldsymbol{\omega_n}\mathbf{t}\} \cdot \mathbf{W}^{-1}\mathbf{q}_o + \mathbf{W} \cdot \left\{\frac{\sin\boldsymbol{\omega_n}\mathbf{t}}{\boldsymbol{\omega_n}}\right\} \cdot \mathbf{W}^{-1}\dot{\mathbf{q}}_o \tag{3.81}$$

3.12.2. Damped Free Vibration

3.12.2.1. Matrix Equation of Motion

The equation of motion which describes damped free vibration follows from Eq. (3.41) with initial conditions $\mathbf{q}(0) = \mathbf{q}_o$ and $\dot{\mathbf{q}}(0) = \dot{\mathbf{q}}_o$

$$\mathbf{B\ddot{q}} + \mathbf{C\dot{q}} + \mathbf{Kq} = \mathbf{0} \tag{3.82}$$

Usually it is convenient to assume that the damping matrix is proportional to either the mass or the stiffness matrix, but the best assumption is that the damping matrix is proportional to both of them (Rayleigh damping), Eq. (3.23), that is

$$\mathbf{C} = \kappa\mathbf{K} + \mu\mathbf{B} \tag{3.83}$$

where μ is the dimensional damping parameter and κ is also a dimensional parameter called the retardation time.

3.12.2.2. Modal Transformation Method

The transformation from the new set of coordinates $\mathbf{r}$ to the generalized coordinates $\mathbf{q}$, such as

$$\mathbf{q} = \mathbf{Wr} \tag{3.84}$$

is substituted into Eq. (3.82). By premultiplying the equation by the transposed modal matrix $\mathbf{W}^T$, and making use of the orthogonal properties of the modal matrix (eigenvectors), the matrix equation has the form

$$\{\mathbf{m}_o\}\ddot{\mathbf{r}} + \{\mathbf{c}_o\}\dot{\mathbf{r}} + \{\mathbf{k}_o\}\mathbf{r} = \mathbf{0} \tag{3.85}$$

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where

$$\{c_o\} = W^T C W \tag{3.86}$$

and in accordance with Eq. (3.84) the initial conditions of Eq. (3.85) are connected with the initial conditions of Eq. (3.82) by relations

$$q_o = W r_o \qquad\qquad \dot{q}_o = W \dot{r}_o \tag{3.87}$$

Eq. (3.85) is the matrix form of a set of uncoupled equations of the Eq. (2.23) type.

3.12.2.3.   Principal Coordinates System

The coordinates by which it is possible to uncouple the MDOF system, described by vector r, are called the principal coordinates system. As it can be seen from Eq. (3.84), the principal coordinates vector r can be achieved from the generalized coordinates vector q with the use of modal matrix W transformation.

The solution in the base of normal coordinates can be achieved by using the solution from Eq. (2.31) separately for each equation from the set of equations (3.85). This solution in matrix notation has the form

$$r(t) = \left\{ e^{-\alpha\omega_n t}\frac{\cos(\omega_d t - \beta)}{\cos\beta} \right\} \cdot r_o + \left\{ e^{-\alpha\omega_n t}\frac{\sin\omega_d t}{\omega_n\cos\beta} \right\} \cdot \dot{r}_o \tag{3.88}$$

where

$$
\begin{aligned}
\left\{ e^{-\alpha\omega_n t}\frac{\cos(\omega_d t - \beta)}{\cos\beta} \right\} &= \mathrm{diag}\left(e^{-\alpha_i\omega_{n,i}t}\frac{\cos(\omega_{d,i} t - \beta_i)}{\cos\beta_i} \right) \\
\left\{ e^{-\alpha\omega_n t}\frac{\sin\omega_d t}{\omega_n\cos\beta} \right\} &= \mathrm{diag}\left(e^{-\alpha_i\omega_{n,i}t}\frac{\sin\omega_{d,i} t}{\omega_{n,i}\cos\beta_i} \right)
\end{aligned}
\tag{3.89}
$$

and after retransformation

$$q(t) = W \cdot \left\{ e^{-\alpha\omega_n t}\frac{\cos(\omega_d t - \beta)}{\cos\beta} \right\} \cdot W^{-1} q_o + W \cdot \left\{ e^{-\alpha\omega_n t}\frac{\sin\omega_d t}{\omega_n\cos\beta} \right\} \cdot W^{-1}\dot{q}_o \tag{3.90}$$

The inverse modal matrix can be calculated without a formal inverse procedure, since from the Orthogonality Principle results

$$W^{-1} = \{m_o\}^{-1} W^T B = \{k_o\}^{-1} W^T K \tag{3.91}$$

Thanks to the assumption Eq. (3.83), a quasi-frequency $\omega_{d,i}$ and a damping ratio α_i can be associated with each natural mode w_i. If the assumption Eq. (3.83) is not fulfilled, the solution is much more complicated. The term $W^T C W$ is not then diagonal and the equations in Eq. (3.85) remain coupled because of the damping matrix. In such a case it is convenient to integrate Eq. (3.82) numerically.

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3.13. Forced Vibration – Response to Harmonic Forces

3.13.1. Direct Method

Let us assume that the force excitation vector in the equation of motion Eq. (3.41) is in the form

$$\mathbf{F}(t) = \mathbf{F}_S \sin \omega t + \mathbf{F}_C \cos \omega t \qquad (3.92)$$

and ω is the angular frequency of harmonic excitation.

The total solution of Eq. (3.41) with harmonic force vector defined by Eq. (3.92) again consists of the complementary solution (transient component of the response) and particular solution (steady-state response) that persist when the transient or the complementary function have disappeared.

The steady-state response of equation of motion Eq. (3.41) could be looked for (Lucky Guess Method or method of undetermined coefficients) also in harmonic form, i.e.

$$\mathbf{q}(t) = \mathbf{q}_S \sin \omega t + \mathbf{q}_C \cos \omega t \qquad (3.93)$$

By substituting Eqs. (3.92) and (3.93) into Eq. (3.41), and then comparing the terms at the sinusoidal and cosinusoidal components of the solution respectively, the algebraic set of equations is achieved

$$(\mathbf{K} - \omega^2 \mathbf{B})\mathbf{q}_S - \omega \mathbf{C}\mathbf{q}_C = \mathbf{F}_S$$
$$\omega \mathbf{C}\mathbf{q}_S + (\mathbf{K} - \omega^2 \mathbf{B})\mathbf{q}_C = \mathbf{F}_C \qquad (3.94)$$

It can now be said that the solution of Eq. (3.41) can exist in the form Eq. (3.94) only if the set of algebraic equations, written in matrix block's form, is fulfilled

$$\begin{bmatrix} \mathbf{K} - \omega^2 \mathbf{B} & -\omega \mathbf{C} \\ \omega \mathbf{C} & \mathbf{K} - \omega^2 \mathbf{B} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{q}_S \\ \mathbf{q}_C \end{bmatrix} = \begin{bmatrix} \mathbf{F}_S \\ \mathbf{F}_C \end{bmatrix} \qquad (3.95)$$

The solution of Eq. (3.95) with respect to vectors $\mathbf{q}_S$ and $\mathbf{q}_C$ makes it possible to determine explicitly the solution of Eq. (3.94).

If the influence of damping is negligible, Eq. (3.95) is reduced to the simple matrix form

$$(\mathbf{K} - \omega^2 \mathbf{B})\mathbf{q}_{S,C} = \mathbf{F}_{S,C} \qquad (3.96)$$

which is valid for both the sinusoidal $\mathbf{q}_S$ and the cosinusoidal $\mathbf{q}_C$ component of solution Eq. (3.94).

From the equation Eq. (3.42), and applying a procedure analogous to the one described previously, the set of algebraic equations has the form

$$\begin{bmatrix} \mathbf{I} - \omega^2\mathbf{DB} & -\omega\mathbf{DC} \\ \omega\mathbf{DC} & \mathbf{I} - \omega^2\mathbf{DB} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{q}_S \\ \mathbf{q}_C \end{bmatrix} = \begin{bmatrix} \mathbf{DF}_S \\ \mathbf{DF}_C \end{bmatrix} \qquad (3.97)$$

If damping matrix is $\mathbf{C} = \mathbf{0}$, the Eq. (3.97) is simplified to the form

$$(\mathbf{I} - \omega^2\mathbf{DB})\mathbf{q}_{S,C} = \mathbf{DF}_{S,C} \qquad (3.98)$$

which is valid for both the sinusoidal $\mathbf{q}_S$ and the cosinusoidal $\mathbf{q}_C$ component of solution Eq. (3.93). The method described above is called the Direct Method because the eigenproblem solution is not necessary. It is also not necessary to assume that the damping matrix is proportional to either the mass or the stiffness matrix or to both of them to achieve the steady-state response of the system. Unfortunately, the size of the system of equations is doubled. Instead of a set of differential equations, a doubled set of coupled algebraic equations must be solved.

Conclusions

Advantages of the Direct Method

in comparison to the Modal Transformation Method approach:

- There is no need to solve the eigenproblem to achieve the steady-state response of the system.

- There is no need to assume that the damping matrix is proportional to either the mass or the stiffness matrix or to both of them to achieve the steady-state response of the system.

Disadvantages of the Direct Method

in comparison to Modal Transformation Method approach:

- It is necessary to solve a doubled set of coupled algebraic equations.

- It is impossible to reduce the base of coordinates used to the determine the solution (the dynamic condensation cannot be performed, see Chapter 3.18)

3.13.2. Modal Transformation Method

If the spectral matrix $\mathbf{\Omega}^2 = \{\omega^2\}$ and the modal matrix $\mathbf{W}$ are known, and if the assumption Eq. (3.83) is valid, and moreover if the damping coefficients specification α_i associated with successive modal forms is known, it is possible to use the same modal transformation, Eq. (3.84), which was applied to the analysis of damped free vibration.

After substituting Eq. (3.84) into Eq. (3.41) and then premultiplicating by $\mathbf{W}^T$, the equation has the form

$$\mathbf{W}^T \mathbf{B}\,\mathbf{W}\ddot{\mathbf{r}} + \mathbf{W}^T \mathbf{C}\,\mathbf{W}\dot{\mathbf{r}} + \mathbf{W}^T \mathbf{K}\,\mathbf{W}\mathbf{r} = \mathbf{W}^T \mathbf{F} = \mathbf{R}(t) \qquad (3.99)$$

Using the Orthogonality Principle Eqs. (3.70), (3.71) and additionally

$$\mathbf{W}^T \mathbf{C}\,\mathbf{W} = \{2\boldsymbol{\alpha}\sqrt{\mathbf{k_o}\mathbf{m_o}}\} = \mathrm{diag}(2\alpha_i \sqrt{k_{o,i}m_{o,i}}) \qquad (3.100)$$

the equation of motion takes the diagonal form

$$\{\mathbf{m_o}\}\ddot{\mathbf{r}} + \{2\boldsymbol{\alpha}\sqrt{\mathbf{k_o}\mathbf{m_o}}\}\dot{\mathbf{r}} + \{\mathbf{k_o}\}\mathbf{r} = \mathbf{R}(t) \qquad (3.101)$$

Matrix Eq. (3.101) is a set of n independent differential equations of SDOF systems. The solution of this system is a superposition of independent solutions of equations of the type

$$m_{o,i}\ddot{r}_i + 2\alpha_i \sqrt{k_{o,i}m_{o,i}}\,\dot{r}_i + k_{o,i}r_i = R_i(t) \qquad (3.102)$$

and could be solved for any kind of excitation.

In the particular situation of a harmonic excitation

$$\mathbf{R}(t) = \mathbf{W}^T (\mathbf{F}_S \sin \omega t + \mathbf{F}_C \cos \omega t) = \mathbf{R}_S \sin \omega t + \mathbf{R}_C \cos \omega t) \qquad (3.103)$$

and

$$\mathbf{r}(t) = \mathbf{r}_S \sin \omega t + \mathbf{r}_C \cos \omega t \qquad (3.104)$$

Using results Eqs. (2.46) and (2.47) it is possible to write

$$\begin{aligned}\mathbf{r}_C &= \{\mathbf{h}_1\}\mathbf{R}_S + \{\mathbf{h}_2\}\mathbf{R}_C, \\ \mathbf{r}_S &= \{\mathbf{h}_1\}\mathbf{R}_C - \{\mathbf{h}_2\}\mathbf{R}_S,\end{aligned} \qquad (3.105)$$

where

$$\begin{aligned}&\{\mathbf{h}_1\} = \mathrm{diag}(h_{11} \quad h_{12} \quad \ldots \quad h_{1d}) \qquad \{\mathbf{h}_2\} = \mathrm{diag}(h_{21} \quad h_{22} \quad \ldots \quad h_{2d}) \\ &h_{1i} = \frac{1}{k_{oi}}\frac{1-\eta_i^2}{(1-\eta_i^2)^2 + (2\alpha_i\eta_i)^2} \qquad h_{2i} = \frac{1}{k_{oi}}\frac{2\alpha_i\eta_i}{(1-\eta_i^2)^2 + (2\alpha_i\eta_i)^2} \\ &\hspace{3cm} \eta_i = \omega/\omega_{ni}\end{aligned} \qquad (3.106)$$

Considering relations $\mathbf{q} = \mathbf{Wr}$ and $\mathbf{R} = \mathbf{W}^T\mathbf{F}$ one can achieve

$$\begin{aligned}\mathbf{q}_S &= \mathbf{W}\{\mathbf{h}_1\}\mathbf{W}^T\mathbf{F}_S + \mathbf{W}\{\mathbf{h}_2\}\mathbf{W}^T\mathbf{F}_c = \mathbf{H}_1\mathbf{F}_S + \mathbf{H}_2\mathbf{F}_C \\ \mathbf{q}_C &= \mathbf{W}\{\mathbf{h}_1\}\mathbf{W}^T\mathbf{F}_C - \mathbf{W}\{\mathbf{h}_2\}\mathbf{W}^T\mathbf{F}_S = \mathbf{H}_1\mathbf{F}_C - \mathbf{H}_2\mathbf{F}_S\end{aligned} \qquad (3.107)$$

or in matrix block form

$$\begin{bmatrix} \mathbf{q}_S \\ \mathbf{q}_C \end{bmatrix} = \begin{bmatrix} \mathbf{H}_1 & \mathbf{H}_2 \\ -\mathbf{H}_2 & \mathbf{H}_1 \end{bmatrix} \cdot \begin{bmatrix} \mathbf{F}_S \\ \mathbf{F}_C \end{bmatrix} \qquad (3.108)$$

where

$$\mathbf{H}_1 = \mathbf{W}\{\mathbf{h}_1\}\mathbf{W}^T \qquad (3.109)$$

$$\mathbf{H}_2 = \mathbf{W}\{\mathbf{h}_2\}\mathbf{W}^T \qquad (3.110)$$

If damping influence is negligible $\mathbf{H}_1 = \mathbf{0}$ the Eq. (3.108) is simplified to the form

$$\mathbf{q} = \mathbf{H}_1\mathbf{F} \qquad (3.111)$$

which is valid for both the sinusoidal and the cosinusoidal component of solution, and

$$h_{1i} = \frac{1}{k_{o,i}(1-\eta_i^2)} \qquad (3.112)$$

Conclusions

Advantages of the Modal Transformation Method
in comparison to the Direct Method approach:

- It is possible to use this approach for another type of forcing excitations, i.e. not for harmonic excitation only.

- The whole analysis can be conducted with the use of SDOF systems only

- It is possible to arbitrarily specify the damping ratios for each mode.

Disadvantages of the Modal Transformation Method
in comparison to the Direct Method approach:

- The eigenproblem analysis must be accomplished

- The assumption is necessary that the damping matrix is proportional to either the mass or the stiffness matrix or to both of them to achieve the steady-state response of the system.

3.14. Dynamic Condensation

It has been seen that the equation of motion of an MDOF system can be uncoupled by using the principal coordinates of the system. By using this device, any single differential equation can be solved for vibration in a given mode at a given frequency. The resultant MDOF system vibration is obtained by the superposition of solutions of all these single differential equations in the base of principal coordinates.

In a discrete mass system, or lumped mass system, with n degrees of freedom, it is rarely necessary to consider all n eigenvalues and eigenvectors. It may be sufficient to know the response in the lowest natural mode, or, perhaps, the first few lowest natural modes. Depending on the significance of the higher modes of vibration, further analyses may be necessary.

Consider the system defined by the Eq. (3.41). Let us assume that the vector $\mathbf{r}_n$ is related to the n first lowest principal coordinates of the vector $\mathbf{r}$ ($n < d$), and that only the n first natural modes described by the n first eigenvectors $\mathbf{W}_n = \begin{bmatrix} \mathbf{w}_1 & \mathbf{w}_2 & \dots & \mathbf{w}_n \end{bmatrix}$ are taken into account – dynamic condensation. If so, the transformation from principal to generalized coordinates is, with accordance to Eq. (3.84), described by the formula

$$\mathbf{q} = \mathbf{W}_n \mathbf{r}_n \tag{3.113}$$

After substitution from Eq. (3.113) and premultiplication by the matrix $\mathbf{W}_n^T$ Eq. (3.112) becomes,

$$\mathbf{W}_n^T \mathbf{B} \mathbf{W}_n \ddot{\mathbf{r}}_n + \mathbf{W}_n^T \mathbf{C} \mathbf{W}_n \dot{\mathbf{r}}_n + \mathbf{W}_n^T \mathbf{K} \mathbf{W}_n \mathbf{r}_n = \mathbf{W}_n^T \mathbf{F}(t) \tag{3.114}$$

or

$$\{\mathbf{m_o}\}_n \ddot{\mathbf{r}}_n + \{2\alpha\sqrt{\mathbf{k_o m_o}}\}_n \dot{\mathbf{r}}_n + \{\mathbf{k_o}\}_n \mathbf{r}_n = \mathbf{R}_n(t) \tag{3.115}$$

Matrix Eq. (3.115) is a set of $n < d$ independent differential equations of SDOF systems. The solution of this system is a superposition of independent solutions of equations of the type

$$m_{o,i} \ddot{r}_i + 2\alpha_i \sqrt{k_{o,i} m_{o,i}} \dot{r}_i + k_{o,i} r_i = R_i(t) \tag{3.116}$$

and could be solved for any kind of excitation. The solutions given in the base of $\mathbf{r}_n$ can be transformed into solutions of the original system in the generalized coordinate base $\mathbf{q}$ given by Eq. (3.113).

The response of the system is then known in the original coordinates in terms of the response in the $n < d$ principle directions only. The solution is now an approximation only; the greater significance of the next higher modes, the closer the approximation will be to the full solution in the base $n = d$. These higher modes may need to be analyzed further.

3.15. Kinematically Forced Vibration

Let us consider a structure whose motion is described by a vector of generalized coordinates $\mathbf{q}$. Let us also assume that there are no external dynamic forces acting on this structure, but the foundation or support of the structure undergoes motion which varies in time. This motion is assumed to be described by the known function of time and by an additional set of coordinates assembled in vector $\mathbf{z}(t)$.

The equation of motion can be written in matrix block form

$$\begin{bmatrix} \mathbf{B}_{qq} & \mathbf{B}_{qz} \\ \mathbf{B}_{zq} & \mathbf{B}_{zz} \end{bmatrix}\begin{bmatrix} \ddot{\mathbf{q}} \\ \ddot{\mathbf{z}} \end{bmatrix} + \begin{bmatrix} \mathbf{C}_{qq} & \mathbf{C}_{qz} \\ \mathbf{C}_{zq} & \mathbf{C}_{zz} \end{bmatrix}\begin{bmatrix} \dot{\mathbf{q}} \\ \dot{\mathbf{z}} \end{bmatrix} + \begin{bmatrix} \mathbf{K}_{qq} & \mathbf{K}_{qz} \\ \mathbf{K}_{zq} & \mathbf{K}_{zz} \end{bmatrix}\begin{bmatrix} \mathbf{q} \\ \mathbf{z} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{Z}(t) \end{bmatrix} \tag{3.117}$$

where $\mathbf{Z}(t)$ is a vector of unknown forces in a base of coordinates $\mathbf{z}(t)$.

The first matrix equation of Eq. (3.117) can be written in the equivalent form

$$\mathbf{B}_{qq}\ddot{\mathbf{q}} + \mathbf{C}_{qq}\dot{\mathbf{q}} + \mathbf{K}_{qq}\mathbf{q} = -\mathbf{B}_{qz}\ddot{\mathbf{z}} - \mathbf{C}_{qz}\dot{\mathbf{z}} - \mathbf{K}_{qz}\mathbf{z} \tag{3.118}$$

Eq. (3.118) can be used to find the solution with respect to $\mathbf{q}(t)$ as a result of kinematic excitation $\mathbf{z}(t)$. If the solution $\mathbf{q}(t)$ is already known, then with the use or equation

$$\mathbf{B}_{zq}\ddot{\mathbf{q}} + \mathbf{B}_{zz}\ddot{\mathbf{z}} + \mathbf{C}_{zq}\dot{\mathbf{q}} + \mathbf{C}_{zz}\dot{\mathbf{z}} + \mathbf{K}_{zq}\mathbf{q} + \mathbf{K}_{zz}\mathbf{z} = \mathbf{Z}(\mathbf{t}) \tag{3.119}$$

it is possible to find the forces $\mathbf{Z}(t)$.

3.16. Variants of Damping Model

In the case of civil engineering and building structures there are usually no lumped dampers but the vibration decays if there are not external excitations. It is also difficult to point to sources of damping, as there seem to be many of them. Therefore, a problem of choosing the damping matrix $\mathbf{C}$ and the generalized forces $\mathbf{Q} = \mathbf{F}_T$ in the base of generalized coordinates $\mathbf{q}$ appears.

Assuming an external character of the resistance to motion, it ought to be assumed that (see also Eq. (2.128))

$$\mathbf{C} = \mu\mathbf{B} \tag{3.120}$$

and then the generalized forces vector $\mathbf{F}_T$ transmitted to the structure has the form

$$\mathbf{F}_T = \mathbf{Kq} = \mathbf{F} - \mathbf{B}(\ddot{\mathbf{q}} - \mu\dot{\mathbf{q}}) \tag{3.121}$$

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In the case of the Voigt-Kelvin rheological model, i.e. when an internal character of resistance to motion is assumed,

$$\mathbf{C} = \kappa \mathbf{K}$$ (3.122)

and then the generalized forces vector $\mathbf{F}_T$ transmitted to the structure has the form

$$\mathbf{F}_T = \mathbf{K}(\mathbf{q} + \kappa \dot{\mathbf{q}}) = \mathbf{F} - \mathbf{B}\ddot{\mathbf{q}}$$ (3.123)

From the earlier analyses (Chapters 2.5.5.5 and 2.5.5.6) it is known that in both cases the assumptions lead to the specification of the damping ratios for each natural mode: $\alpha_i = \mu / 2\omega_{n,i}$ and $\alpha_i = \kappa \omega_{n,i} / 2$ respectively. Experimental investigations do not validate either of these hypotheses of damping, i.e. these methods of specifying damping coefficients.

The Rayleigh model (Chapter 2.5.5.7) takes into account both the viscous damping representing an environmental damping by external forces acting on the structure (external damping), and the damping due to the inherent properties of the material of the structure (internal damping). Thus, the damping matrix is assumed to have the form

$$\mathbf{C} = \kappa \mathbf{K} + \mu \mathbf{B}$$ (3.124)

and then the generalized forces vector $\mathbf{F}_T$ transmitted to the structure has the form

$$\mathbf{F}_T = \mathbf{K}(\mathbf{q} + \kappa \dot{\mathbf{q}}) = \mathbf{F} - \mathbf{B}(\ddot{\mathbf{q}} - \mu \dot{\mathbf{q}})$$ (3.125)
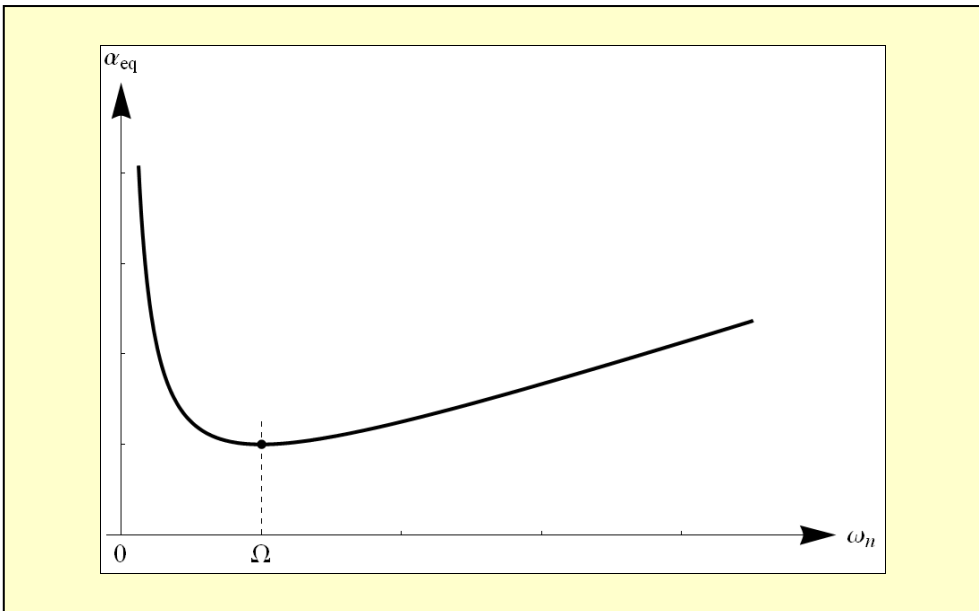

Fig. 3.7 Dependence of equivalent damping ratio on the natural frequency value

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In this case a non-dimensional damping ratio, Fig. 3.7, given by the formula below, is assigned to each mode of vibration

$$\alpha_i = \frac{\kappa\omega_{n,i}}{2} + \frac{\mu}{2\omega_{n,i}} \qquad (3.126)$$

For the frequency $\Omega = \sqrt{\mu/\kappa}$ the damping ratio reaches its minimum $\alpha_{\min} = \sqrt{\mu\kappa}$. If the frequency Ω is recognized as the most important one, the minimal coefficient ratio $\alpha = \alpha_{\min}$ will be assonated to it. This assumption leads to the result

$$\mu = \alpha\,\Omega \qquad\qquad \kappa = \alpha/\Omega \qquad (3.127)$$

It is also possible to choose two important natural frequencies $\omega_{n,i}$, $\omega_{n,j}$ and to require that the condition $\alpha_i = \alpha_j = \alpha$ be fulfilled for two equations received after substituting these conditions into Eq. (3.126). The solution of these two equations gives two parameters: μ and κ .

It is important to note that the problems with specifying the damping matrix decrease when the modal transformation procedure is used. In this case it is possible to arbitrarily specify the damping ratios α_i for each mode. It is also possible to assume $\alpha_i = \alpha_j = \alpha = const$, which is tantamount to the assumption that

$$\mathbf{C} = \mathbf{BW}\{2\alpha\,\omega_{\mathbf{n}}/\mathbf{m_o}\}\mathbf{W}^T\mathbf{B} + \mathbf{KW}\{2\alpha/\omega_{\mathbf{n}}\mathbf{k_o}\}\mathbf{W}^T\mathbf{K} \qquad (3.128)$$

but then the problem of generalized transmitted force remains undetermined.

In the analysis of the steady-state harmonic vibration of discrete systems it is recommended to assume the Voigt-Kelvin model of damping with the parameter $\kappa = 2\alpha/\omega$, where α is the damping ratio appropriate for a given type of structure and ω is the angular frequency of harmonic excitation. Then

$$\mathbf{C} = \frac{2\alpha}{\omega}\mathbf{K} \qquad (3.129)$$

$$\mathbf{F}_T = \mathbf{K}(\mathbf{q} + \frac{2\alpha}{\omega}\dot{\mathbf{q}}) = \mathbf{F} + \omega^2\mathbf{Bq} \qquad (3.130)$$

This assumption can be interpreted in the principal coordinate system as

$$c_{o,i} = \frac{2\alpha}{\omega} k_{o,i} \qquad (3.131)$$

$$\alpha_i = \frac{c_{o,i}}{2\sqrt{k_{o,i} m_{o,i}}} = \frac{\alpha \omega_{n,i}}{\omega} = \frac{\alpha}{\eta_i} \qquad (3.132)$$

It can be seen that in the resonance zone $\omega \approx \omega_{n,i}$, in which the damping is significant, $\alpha_i \approx \alpha$. However, outside the resonance zone, the damping ratio $\alpha_i = \alpha/\eta_i$ is different than α. Even so, as the influence of small damping is negligible, this is of no great importance. Thus, the assumptions from Eqs. (3.129) and (3.130) are practically tantamount to the hypothesis of the determined decrement of damping. It is important to notice that this assumption can only be made for steady-state harmonic solutions of lightly damped systems.

The vibration analysis is simplified significantly when the resistance to motion is negligible. In that case

$$\mathbf{C = 0} \qquad\qquad \mathbf{F}_T = \mathbf{Kq} = \mathbf{F} - \mathbf{B\ddot{q}} \qquad (3.133)$$

In addition, if the flexibility matrix $\mathbf{D}$ is used, it is possible to write the equation of motion in the base of generalized forces $\mathbf{F}_T = \mathbf{Q}$ as follows

$$\mathbf{BD\ddot{Q} + Q = F} \qquad (3.134)$$

In the case of harmonic excitation, instead of Eq. (3.98), the following equation is valid

$$(\mathbf{I} - \omega^2 \mathbf{DB})\mathbf{Q}_{S,C} = \mathbf{F}_{S,C} \qquad (3.135)$$

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CHAPTER 4

PLANE BEAMS, FRAMES, TRUSSES AND FOUNDATIONS

4. Plane Beams, Frames, Trusses and Foundations

"The first step in analyzing any physical structure is to represent it by a mathematical model which will have essentially the same dynamic behavior. A suitable number and distribution of masses, springs, and dampers must be chosen, and the input forces or foundation motions must be defined. The model should have sufficient degrees-of-freedom to determine the modes which will have significant response to the exciting force or motion.

The properties of a system that must be known are the natural frequencies ω_n, the normal mode shapes, … the damping of the respective modes, and the mass distribution m_j. The detailed distribution of stiffness and damping of a system are not used directly but rather appear indirectly as the properties of the respective modes. The characteristic properties of the modes may be determined experimentally as well as analytically.", [1].

4.1. Plane Beams and Frames

Let us consider plane beams or frames systems, constructed with weightless plain bars. The axes of the bars intersect in geometrical nodes. There are lumped masses and rigid bodies in the structure. In the mass center of each lumped mass, two local translational cartesian coordinates are assumed to exist, and in the mass center of each rigid body an additional rotational coordinate must be assumed. The whole set of coordinates is written as a vector

$$\mathbf{u} = [u_1 \quad u_2 \quad u_3 \quad \ldots]^T \tag{4.1}$$

As it was shown in Chapter 3.4.2.2, the inertia matrix in the local coordinates base of mass centers has a diagonal form, Eq. (3.11)

$$\{\mathbf{m}\} = diag(m_1 \quad m_2 \quad m_3 \quad \ldots) = \begin{bmatrix} m_1 & 0 & 0 & \cdots \\ 0 & m_2 & 0 & \cdots \\ 0 & 0 & m_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \tag{4.2}$$

where m_j are the masses *sensu stricto* (associated with translational coordinates) or mass moments $m_j = J_j$ (associated with rotational coordinates). The generalized coordinates vector

$$\mathbf{q} = [q_1 \quad q_2 \quad q_3 \quad \ldots]^T \tag{4.3}$$

can be transformed to the local coordinates vector $\mathbf{u}$ with accordance to Eq. (3.9)

$$\mathbf{u} = \mathbf{A}_m \mathbf{q} \tag{4.4}$$

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The inertia matrix in the generalized coordinates base has the form determined by Eq. (3.14), i.e.

$$\mathbf{B} = \mathbf{A}_m^T \cdot \{\mathbf{m}\} \cdot \mathbf{A}_m \tag{4.5}$$

The structure can be excited by the generalized forces, Eq. (3.35)

$$\mathbf{F} = \mathbf{A}_f^T \mathbf{P} \tag{4.6}$$

The next calculations of the dynamic analysis of such structures could be realized with the use of the Force or the Displacement Method.

4.1.1. Displacement Method

4.1.1.1. Kinematically (Geometrically) Indeterminate Structures

A rigid body system consists of a number of rigid bodies in space. The degrees of freedom of these independent rigid bodies can be removed by adding kinematic constraints, thanks to which the number of degrees of freedom is reduced.

A system is kinematically indeterminate if the number of unknown node/joint displacements that are needed to describe the displaced shape of the structure is greater than zero.

4.1.1.2. Degree of Kinematic (Geometric) Indeterminacy

The degree of kinematic (geometric) indeterminacy is the number of kinematic constraints necessary to achieve the kinematic (geometrical) determinacy of the system. It is described by the formula

$$n_g = n_\Delta + n_\varphi \tag{4.7}$$

where

n_φ – the number of rotational constraints necessary to obtain geometric
determinacy from the point of view of the Displacement Method

n_Δ – the number of translational constraints it is necessary to add to the kinematic
chain in order to obtain a geometrically stable and statically determinate truss

and

$$n_\Delta = 2w - (p + r) \tag{4.8}$$

where

w – number of truss hinges in a kinematic chain
p – number of members in a kinematic chain
r – number of supporting constraints (links) in a kinematic chain

4.1.1.3. Degree of Kinematic (Geometric) Indeterminacy in a Dynamic Sense

The degree of kinematic (geometric) indeterminacy in a dynamic sense n_{gd} is defined as the number of degrees of kinematic indeterminacy n_g reduced by the number of degrees of freedom which are the dynamic generalized coordinate d. Therefore, the number n_{gd} can be interpreted as the number of additional non-dynamic information, necessary only due to static solution of the structure in the Displacement Method sense (Fig. 4.2).

$$n_{gd} = n_g - d \tag{4.9}$$

4.1.1.4. Stiffness Matrix

4.1.1.4.1. Kinematic Coordinates Vector in the Dynamic Sense

The kinematic coordinates vector in the dynamic sense will be expressed as $\mathbf{x}$. The vector $\mathbf{x}$ is the displacement vector of those points of the structure which are neither mass points nor points of fixing the rigid bodies to the structure, but which are necessary to kinematically determine the structure from the Displacement Method point of view.

4.1.1.4.2. Expanded Base of Kinematic Coordinates

The expanded base of kinematic coordinates is defined as the vector

$$\hat{\mathbf{q}} = \begin{bmatrix} \mathbf{q} \\ \mathbf{x} \end{bmatrix} \tag{4.10}$$

The displacement vector in the expanded base of kinematic coordinates consists of two subvectors:
- subvector

$$\mathbf{q} = [q_1 \quad \dots \quad q_d]^T \tag{4.11}$$

which is the generalized coordinates vector whose dimension is equal to the number of generalized coordinates d and
- subvector

$$\mathbf{x} = [x_1 \quad \dots \quad x_{n_{gd}}]^T \tag{4.12}$$

which is the kinematic (geometric) coordinates vector in the dynamic sense, whose dimension is n_{gd}.

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4.1.1.4.3. Stiffness Matrix in an Expanded Base of Coordinates

The stiffness matrix in an expanded base of coordinates is defined as

$$\hat{\mathbf{K}} = \begin{bmatrix} \mathbf{K}_{qq} & \mathbf{K}_{qx} \\ \mathbf{K}_{xq} & \mathbf{K}_{xx} \end{bmatrix} \tag{4.13}$$

$$\mathbf{K}_{qx} = \mathbf{K}_{xq}^{T} \tag{4.14}$$

where: $\quad \dim \mathbf{K}_{qq} = d \times d$, $\dim \mathbf{K}_{qx} = d \times n_{gd}$, $\dim \mathbf{K}_{xq} = n_{gd} \times d$, $\dim \mathbf{K}_{xx} = n_{gd} \times n_{gd}$

4.1.1.5. Static Condensation

In order to avoid terminological misunderstandings, it is necessary to mention that the term "Static Condensation" used in this book has a different meaning than the same term as used by another authors. Contrary to the definition of Static Condensation formulated for example by M. Paz and W. Leigh in [1], that "...Static Condensation Method...is only approximate and may produce relatively large errors in the results when applied to dynamic problems", the Static Condensation Method proposed here, devised by J. Langer in [4], gives exact results when applied to dynamic problems. The main difference from the method proposed in [1] is that, in the method presented here, there are no primary and secondary (less important) degrees of freedom. Degrees of freedom are here divided into Lagrange's generalized coordinates $\mathbf{q}$ and kinematic (geometric) degrees of freedom $\mathbf{x}$ which do not describe displacements of the mass points, but are necessary if the Primary Structure of the Displacement Method is to be kinematically determined. After the solution of the dynamic problem in the base of generalized coordinates $\mathbf{q}$ it is always possible to find the kinematic degrees of freedom $\mathbf{x}$ from Eq. (4.16).

The equilibrium conditions of the Displacement Method in the expanded base of coordinates $\hat{\mathbf{q}}$, Eq. (4.10), has the form

$$\mathbf{K}_{xq}\mathbf{q} + \mathbf{K}_{xx}\mathbf{x} = \mathbf{0} , \tag{4.15}$$

therefore from here

$$\mathbf{x} = -\mathbf{K}_{xx}^{-1}\mathbf{K}_{xq}\mathbf{q} \tag{4.16}$$

From the identity

$$\mathbf{K}_{qq}\mathbf{q} + \mathbf{K}_{qx}\mathbf{x} = \mathbf{K}\,\mathbf{q} \tag{4.17}$$

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after substituting Eq. (4.16) into (4.17), one can achieve the stiffness matrix in the base of generalized coordinates $\mathbf{q}$ from formula

$$\mathbf{K} = \mathbf{K}_{qq} - \mathbf{K}_{qx}\mathbf{K}_{xx}^{-1}\mathbf{K}_{xq} \qquad (4.18)$$

For an SDOF system the kinematic indeterminacy degree $n_{gd} = 1$, from Eq. (4.18) one can find the equivalent stiffness coefficient

$$k = k_{qq} - \frac{k_{qx}k_{xq}}{k_{xx}} \qquad (4.19)$$

4.1.2. Force Method

4.1.2.1. Degree of Static Indeterminacy

4.1.2.1.1. Statically Indeterminate Structures

A system is statically indeterminate if the number of unknown member forces and, optionally, reactions in the system, is greater than the number of independent, non-trivial equilibrium equations available for determining these unknown forces.
In general, the static indeterminacy of structural systems depends not only on their external supports, but on their internal structures as well.

4.1.2.1.2. Degree of Static Indeterminacy

The degree of static indeterminacy of a system (number of redundants, or number of hyperstatics) is $n_h = n_M - n_N$ where

n_M is the number of unknown member forces, and optionally, reactions in the system;

n_N is the number of independent, non-trivial equilibrium equations available for determining these n_M unknown forces.

In practice, it is more convenient to determine the degree of static indeterminacy of a plane system by using the formula

$$n_h = e - 3t \qquad (4.20)$$

or in space

$$n_h = e - 6b \qquad (4.21)$$

where

e is the number of constraints in the system,

t is the number of rigid bodies in the 2D system,

b is the number of rigid bodies in the 3D system

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4.1.2.2. Flexibility Matrix

4.1.2.2.1. Redundant (Hyperstatic) Forces Vector

The redundant forces vector of the Force Method will be expressed as vector $\mathbf{X}$ whose dimension is n_h.

4.1.2.2.2. Independent Characteristic Forces Vector

The independent characteristic forces vector $\mathbf{Q}$ is a vector of unknown internal forces transmitted from the mass to the structure. The vector $\mathbf{Q}$ is in accordance with the generalized coordinates vector $\mathbf{q}$. The dimension of the vector $\mathbf{Q}$ is equal to the number of generalized coordinates d.

4.1.2.2.3. Primary Structure (Scheme) of Force Method

To create a Primary Structure of the Force Method it is necessary to make the assumption that a dynamic force Q_i acts in the place and the direction of the generalized coordinate q_i, and a hyperstatic force X_i acts in the place and the direction of the restraints x_i (Fig. 4.3). The dimension of vectors $\mathbf{q}$ and $\mathbf{Q}$ is d, but the dimension of vectors $\mathbf{x}$ and $\mathbf{X}$ is n_h.

4.1.2.2.4. Expanded Base of Kinetic Coordinates

The vector of dynamic forces in the expanded base of kinetic coordinates $\widehat{\mathbf{Q}}$

$$\widehat{\mathbf{Q}} = \begin{bmatrix} \mathbf{Q} \\ \mathbf{X} \end{bmatrix} \tag{4.22}$$

consists of two subvectors:
- subvector

$$\mathbf{Q} = [\, Q_1 \quad \ldots \quad Q_d \,]^T \tag{4.23}$$

whose dimension is equal to d and
- subvector

$$\mathbf{X} = [\, X_1 \quad \ldots \quad X_{n_h} \,]^T \tag{4.24}$$

whose dimension is equal to n_h

4.1.2.2.5. Flexibility Matrix in Expanded Base of Coordinates

The flexibility matrix in the expanded base of coordinates is defined

$$\hat{\mathbf{D}} = \begin{bmatrix} \mathbf{D}_{qq} & \mathbf{D}_{qx} \\ \mathbf{D}_{xq} & \mathbf{D}_{xx} \end{bmatrix} \tag{4.25}$$

where

$$\mathbf{D}_{qx} = \mathbf{D}_{xq}^{T} \tag{4.26}$$

and

$$\dim \mathbf{D}_{qq} = d \times d , \ \dim \mathbf{D}_{qx} = d \times n_h , \ \dim \mathbf{D}_{xq} = n_h \times d , \ \dim \mathbf{D}_{xx} = n_h \times n_h$$

4.1.2.3. Static Condensation

After operations analogical to the ones in the case of the stiffness matrix (Chapter 4.1.1.6), it is possible to find the flexibility matrix in the base of generalized forces $\mathbf{Q}$, from formula

$$\mathbf{D} = \mathbf{D}_{qq} - \mathbf{D}_{qx}\mathbf{D}_{xx}^{-1}\mathbf{D}_{xq} \tag{4.27}$$

and the redundant forces vector from

$$\mathbf{X} = -\mathbf{D}_{xx}^{-1}\mathbf{D}_{xq}\mathbf{Q} \tag{4.28}$$

The dimensions of matrices $\mathbf{K}$ and $\mathbf{D}$ are the same and are equal to the number of generalized coordinates d, and of course

$$\mathbf{K} = \mathbf{D}^{-1} \tag{4.29}$$

For an SDOF system the static indeterminacy degree $n_h = 1$, Eq. (4.29) has the form

$$\delta = \delta_{qq} - \frac{\delta_{qx}\delta_{xq}}{\delta_{xx}} \tag{4.30}$$

4.1.3. Choice of the Method

In a given case, either the Displacement Method or the Force Method may be the most convenient for formulating the matrix equation of motion, as it will give rise to a lesser number of additional hyperstatic or hyperkinematic unknowns. In order to choose, which method is the most convenient, it is sufficient to check one simple criterion. Namely, one has to determine which is the greater: the degree of kinematic indeterminacy in the dynamic sense n_{gd} (calculated from formula Eq. (4.9)) or the degree of static indeterminacy of the system n_h (calculated from formula Eq. (4.20)).

Choice of the Method:

- If $n_h > n_{gd}$ the Displacement Method should be chosen.
- If $n_h < n_{gd}$ the Force Method should be chosen.
- If $n_h = n_{gd}$ either of these methods can be chosen.

4.1.4. Forced Vibration

When the vector of generalized forces and the inertia, damping, stiffness or flexibility matrix are achieved, it is enough to write the equation of motion, Eq. (3.41) or (3.42), and solve it using the algorithms which are presented in Chapter 3. There are three possibilities, which will be considered below.

4.1.4.1. Modal Transformation Method

If the inertia and the stiffness/flexibility matrices are known, the eigenproblem may be solved, see Chapter 3.8. Next, using the Modal Transformation Method, see Chapter 3.13.2, the equations of motion can be decoupled, Eq. (3.101). Then, the system of independent equations, of the Eq. (3.102) type, can be solved separately in the principal coordinates base. The final solution is achieved with the use of these separate solutions and the superposition method. The solution in the generalized coordinates base may be obtained with the use of the modal transformation, Eq. (3.84). The dynamic condensation approach, Chapter 3.14, can also be used.

Conclusions

Advantages of the Modal Transformation Method of
solving the equations of motion for beam or frame structures:
- The possibility to use this approach for other types of forcing excitations, i.e. not for harmonic excitation only.
- The possibility to reduce the base of principle coordinates used to determine the final solution in the generalized coordinates base (dynamic condensation can be carried out).
- The possibility to conduct the whole analysis with the use of SDOF systems only.
- The possibility to arbitrarily specify the damping ratios for each mode.

4.1.4.2. Direct Method

When only the harmonic excitation forces act in the system, the solution of the equation of motion is usually realized with the use of the Direct Method, Chapter 3.13.1. Analyzing harmonically forced vibration, and taking into account damping, the displacement steady-state response can be determined from Eqs. (3.95) or (3.97). The independent characteristic forces can then be evaluated from equations

$$\begin{aligned} \mathbf{Q}_S &= \mathbf{F}_S + \omega^2 \mathbf{B}\mathbf{q}_S \\ \mathbf{Q}_C &= \mathbf{F}_C + \omega^2 \mathbf{B}\mathbf{q}_C \end{aligned} \tag{4.31}$$

Bending moments M_S, M_C, shear forces T_S, T_C and axial (normal) forces N_S, N_C can be then calculated, and their diagrams can be drawn. It is also possible to calculate the amplitudes of the bending moments, shear forces and axial forces using the formulas

$$\begin{aligned} \operatorname{am} M &= \sqrt{M_S^2 + M_C^2} \\ \operatorname{am} T &= \sqrt{T_S^2 + T_C^2} \\ \operatorname{am} N &= \sqrt{N_S^2 + N_C^2} \end{aligned} \tag{4.32}$$

The amplitudes of the bending moments diagram, the amplitudes of the shear forces diagram and the amplitudes of the normal forces diagram are the dynamic envelopes of these section forces, i.e. the envelopes with respect to time. The amplitudes of shear forces and the amplitudes of normal forces diagrams are usually constant between points of concentrated force application. The amplitude of bending moments diagrams are usually curvilinear between points of concentrated force application, except for the situation when the diagrams M_S, M_C are proportional.

If the damping influence is negligible, the state of displacement can be calculated from Eqs. (3.96) or (3.98). The independent characteristic forces can then be evaluated from equations, Eq. (4.31), or independently of displacements, from Eq. (3.135) directly, separately for $\mathbf{Q}_S$ and $\mathbf{Q}_C$.

Notice

The static condensation procedure, which was described in Chapter 4.1.1.6, can be avoided if the generalized coordinate base is assumed to be an already expanded one ($n > d$, $\det\mathbf{B} = 0$), i.e. all the equations of motion are formulated in an already expanded generalized coordinate base. This is possible in consideration of the homogeneous character of the unknowns $\mathbf{q}$ and $\mathbf{x}$. Unfortunately, the number of the dynamic unknowns increases. The matrix equation of motion then has the form

$$\begin{bmatrix} \mathbf{B}_{qq} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}\begin{bmatrix} \ddot{\mathbf{q}} \\ \ddot{\mathbf{x}} \end{bmatrix} + \begin{bmatrix} \mathbf{C}_{qq} & \mathbf{C}_{qx} \\ \mathbf{C}_{xq} & \mathbf{C}_{qx} \end{bmatrix}\begin{bmatrix} \dot{\mathbf{q}} \\ \dot{\mathbf{x}} \end{bmatrix} + \begin{bmatrix} \mathbf{K}_{qq} & \mathbf{K}_{qx} \\ \mathbf{K}_{xq} & \mathbf{K}_{qx} \end{bmatrix}\begin{bmatrix} \mathbf{q} \\ \mathbf{x} \end{bmatrix} = \begin{bmatrix} \mathbf{F} \\ \mathbf{0} \end{bmatrix}$$

If the Force Method is applied, the assumption that coordinate base is already expanded is impossible, because of the non-homogenous character of unknowns $\mathbf{q}$ and $\mathbf{X}$.

Conclusions

Advantages of the Direct Method of solving the equations of motion for beam or frame structures:

- The eigenproblem solution is not necessary to achieve the steady-state response of the system.
- The assumption that the damping matrix is proportional to either the mass or the stiffness matrix or to both of them is not necessary to achieve the steady-state response of the system.

Disadvantages of the Direct Method of solving the equations of motion for beam or frame structures:

- It is necessary to solve a doubled set of coupled algebraic equations.
- It is impossible to reduce the coordinates base used to determine the solution (the dynamic condensation cannot be performed).

4.1.4.3. Numerical Integration

The third possibility to solve the matrix equation of motion which describes the forced vibration of the beam or frame is to numerically integrate this equation with the use of the appropriate numerical method.

Conclusions

Advantages of the numerical integration method of solving the equations of motion for beam or frame structures:

- The eigenproblem solution is not necessary to achieve the solution of the system.

- The assumption that the damping matrix is proportional to either the mass or the stiffness matrix or to both of them is not necessary.

- It is possible to achieve the solution for any type of forcing excitations.

- Solutions can be obtained not only for steady-state response.

Disadvantages of the numerical integration method of solving the equations of motion for beam or frame structures:

- The solutions are achieved in a non-analytical form.

- The solutions are very time-consuming.

- Special algorithms should be applied.

Illustrative Example 4.1

Let us consider a statically and kinematically indeterminate plane frame structure. An example of the dynamic scheme of a frame structure is shown in Fig. 4.1.
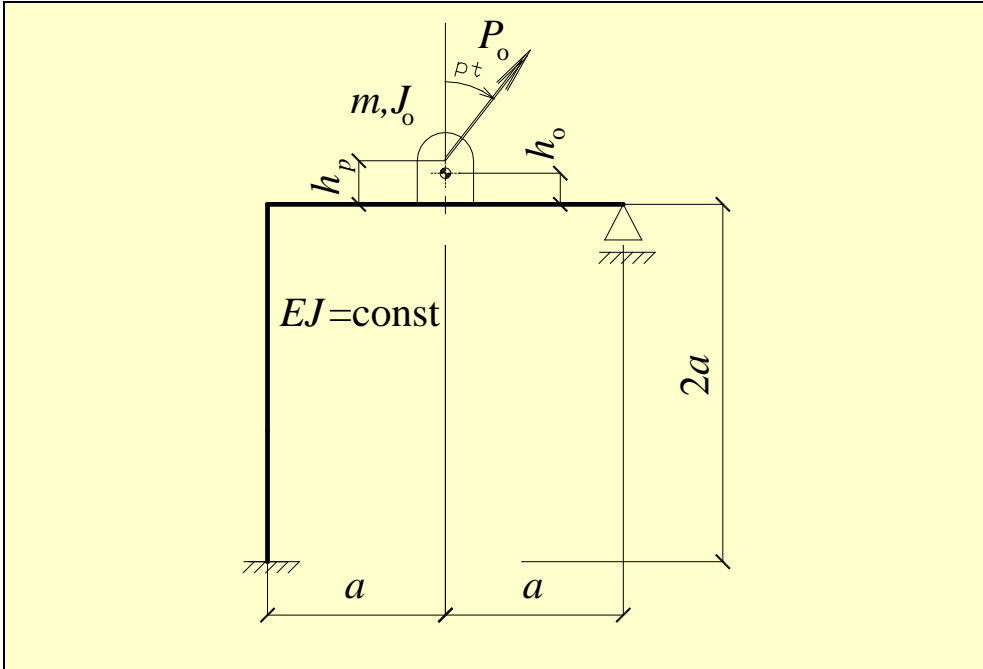

Fig. 4.1 Dynamic scheme of the frame

where

$a = 3\,\text{m}$	$m = 500\,kg$
$E = 200\,\text{GPa}$	$J_O = 20.8\,\text{kgm}^2$
$I = 9800\,\text{cm}^4\,(\text{I}\,300)$	$h_O = 0.25\text{m}$
$EI = \text{const}$	$h_P = 0.40\text{m}$
$EA = \infty$	$P_O = 1\,\text{kN}$
$GA = \infty$	$\omega = 30\,\text{rad/s}$

The number of degrees of freedom (Fig. 4.2)

$$d = d_\Delta + d_\varphi = 2 + 1 = 3$$

The degree of static indeterminacy is (Fig. 4.2)
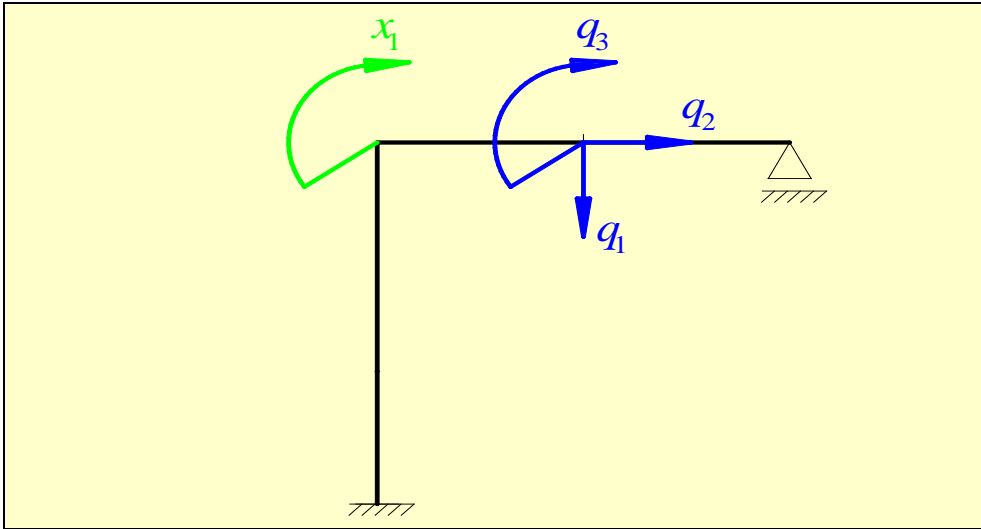
$$n_h = e - 3t = 4 - 3 \cdot 1 = 1$$

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Fig. 4.2 Force Method scheme of coordinates

The degree of kinematic (geometric) indeterminacy is (Fig. 4.2)

$$n_g = n_\Delta + n_\varphi = 2 + 2 = 4$$

The degree of kinematic (geometric) indeterminacy in a dynamic sense
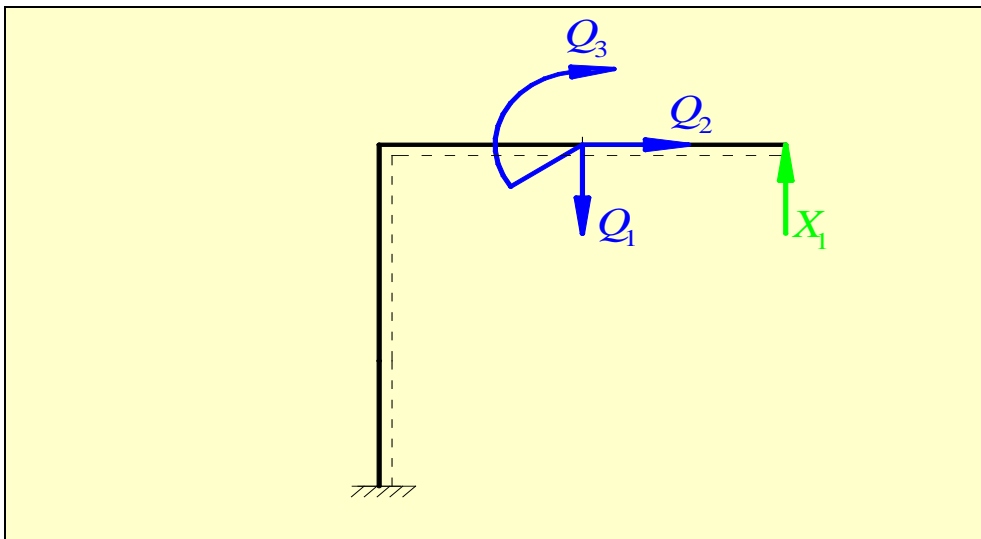
$$n_{gd} = n_g - d = 4 - 3 = 1$$

Fig. 4.3 Displacement Method scheme of coordinates

Because $n_h = n_{gd}$ either of the Displacement Method or the Force Method can be chosen. The Displacement Method has been chosen arbitrary.

The generalized coordinates vector **q** is

$$\mathbf{q} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}$$

The **x** vector consists of one element only

$$\mathbf{x} = [x_1]$$

The expanded base of kinematic coordinates is defined as the vector

$$\hat{\mathbf{q}} = \begin{bmatrix} \mathbf{q} \\ \mathbf{x} \end{bmatrix} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ x_1 \end{bmatrix}$$

The stiffness matrix in an expanded base of coordinates is defined as

$$\hat{\mathbf{K}} = \begin{bmatrix} \mathbf{K}_{qq} & \mathbf{K}_{qx} \\ \mathbf{K}_{xq} & \mathbf{K}_{xx} \end{bmatrix} = \begin{bmatrix} \dfrac{15\,EI}{l^3} & 0 & \dfrac{-3\,EI}{l^2} & \dfrac{-6\,EI}{l^2} \\ 0 & \dfrac{3\,EI}{2l^3} & 0 & \dfrac{-3\,EI}{2l^2} \\ \dfrac{-3\,EI}{l^2} & 0 & \dfrac{7\,EI}{l} & \dfrac{2\,EI}{l} \\ \dfrac{-6\,EI}{l^2} & \dfrac{-3\,EI}{2l^2} & \dfrac{2\,EI}{l} & \dfrac{6\,EI}{l} \end{bmatrix}$$

where

$$\mathbf{K}_{qx} = \mathbf{K}_{xq}^T = \begin{bmatrix} \dfrac{-6\,EI}{l^2} \\ \dfrac{-3\,EI}{2l^2} \end{bmatrix}$$

$$\mathbf{K}_{xx} = \begin{bmatrix} \dfrac{6\,EI}{l^2} \end{bmatrix}$$

Static Condensation yields, Eq. (4.18), the stiffness matrix in generalized coordinates base

$$\mathbf{K} = \mathbf{K}_{qq} - \mathbf{K}_{qx}\mathbf{K}_{xx}^{-1}\mathbf{K}_{xq} =$$

$$= \begin{bmatrix} \dfrac{6EI}{l^3} & -\dfrac{3EI}{2l^3} & -\dfrac{EI}{l^2} \\[2mm] -\dfrac{3EI}{2l^3} & \dfrac{9EI}{8l^3} & \dfrac{EI}{2l^2} \\[2mm] -\dfrac{EI}{l^2} & \dfrac{EI}{2l^2} & \dfrac{19EI}{3l} \end{bmatrix} = \begin{bmatrix} 6.53\cdot10^6 & -1.09\cdot10^6 & -2.18\cdot10^6 \\ -1.09\cdot10^6 & 0.82\cdot10^6 & 1.09\cdot10^6 \\ -2.18\cdot10^6 & 1.09\cdot10^6 & 41.14\cdot10^6 \end{bmatrix}$$

Mass matrix can be achieved with the use of transformation from local to generalized coordinates, Fig. 4.4.
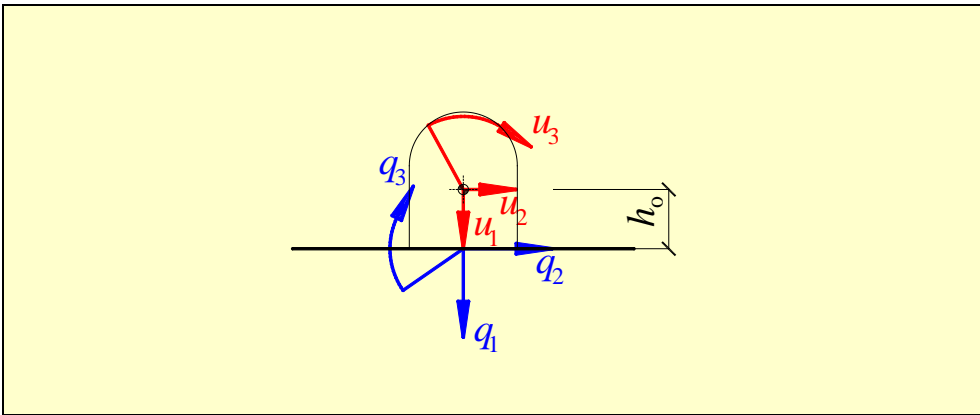

Fig. 4.4 Generalized and local coordinates associated with mass center

The local coordinates transformation to generalized coordinate may be formulated as

$$\mathbf{u} = \mathbf{A}_m\mathbf{q} \quad \rightarrow \quad \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & h_o \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}$$

The inertia (mass) matrix in the local coordinates base of mass centers has a diagonal form, Eq. (3.11)

$$\{\mathbf{m}\} = diag(m \quad m \quad J_O) = \begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & J_O \end{bmatrix}$$

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The inertia (mass) matrix in the generalized coordinates base has the form

$$\mathbf{B} = \mathbf{A}_m^T \cdot \{\mathbf{m}\} \cdot \mathbf{A}_m = \begin{bmatrix} m & 0 & 0 \\ 0 & m & mh_O \\ 0 & mh_O & J_O + mh_O^2 \end{bmatrix} = \begin{bmatrix} 500 & 0 & 0 \\ 0 & 500 & 125 \\ 0 & 125 & 52.08 \end{bmatrix}$$

The structure can be excited by the generalized forces, Eq. (3.35)

$$\mathbf{F} = \begin{bmatrix} -P_O \cos pt \\ P_O \sin pt \\ P_O h_p \sin pt \end{bmatrix} = \begin{bmatrix} 0 \\ P_O \\ P_O h_p \end{bmatrix} \sin pt + \begin{bmatrix} -P_0 \\ 0 \\ 0 \end{bmatrix} \cos pt = \mathbf{F}_S \sin pt + \mathbf{F}_C \cos pt$$

Finally, matrix equation of motion has the form

$$\mathbf{B}\ddot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{F}(t)$$

The eigenproblem formulation can be written down as

$$\mathbf{B}\ddot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{0}$$

and

$$(\mathbf{K} - \omega^2 \mathbf{B})\mathbf{q} = \mathbf{0}$$

The solution of this algebraic set of equation leads to eigenvalues and natural frequencies

$$\begin{aligned} \omega_1 &= 34.91[\text{rad/s}] & \rightarrow & & f_1 &= 5.56[\text{Hz}] \\ \omega_2 &= 115.2[\text{rad/s}] & \rightarrow & & f_2 &= 18.3[\text{Hz}] \\ \omega_3 &= 1401[\text{rad/s}] & \rightarrow & & f_3 &= 223[\text{Hz}] \end{aligned}$$

and eigenvectors and modal matrix

$$\mathbf{W} = [\mathbf{w}_1 \quad \mathbf{w}_2 \quad \dots \quad \mathbf{w}_n] = \begin{bmatrix} 0.178 & 1 & -0.002 \\ 1 & -0.193 & -0.250 \\ -0.014 & 0.050 & 1 \end{bmatrix}$$
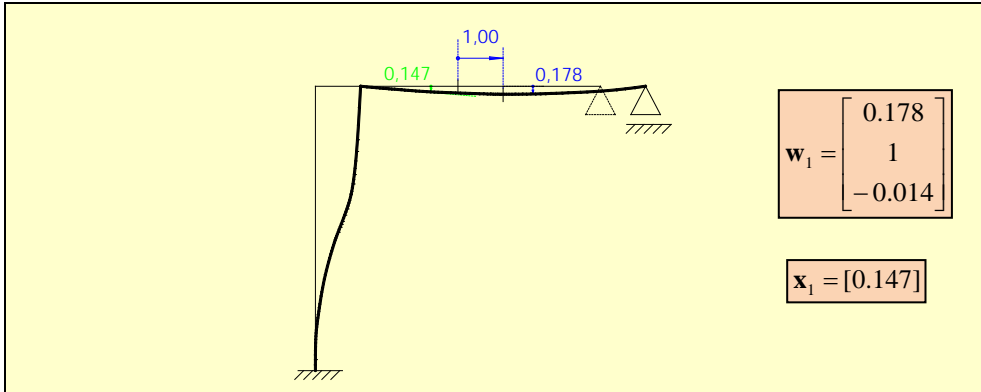
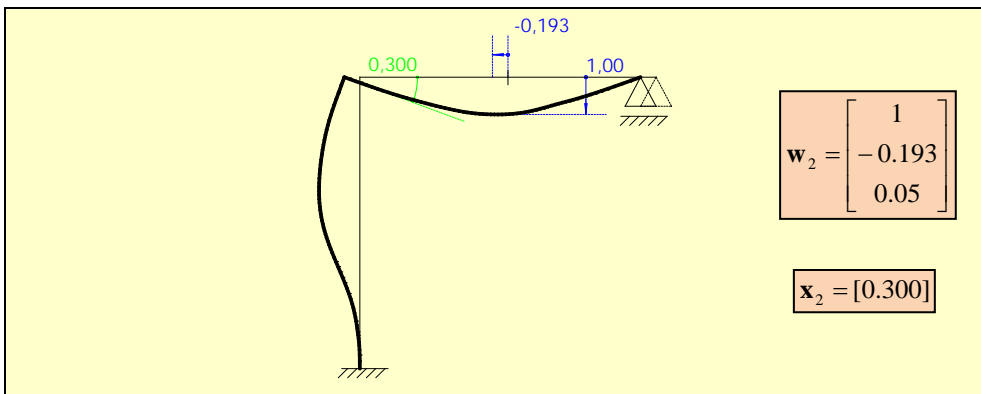$$\mathbf{x}_1 = [0.147] \qquad \mathbf{x}_2 = [0.300] \qquad \mathbf{x}_3 = [-0.355]$$

Natural modes of vibration are shown below

- First mode of vibration

$$\mathbf{w}_1 = \begin{bmatrix} 0.178 \\ 1 \\ -0.014 \end{bmatrix}$$

$$\mathbf{x}_1 = [0.147]$$

- Second mode of vibration

$$\mathbf{w}_2 = \begin{bmatrix} 1 \\ -0.193 \\ 0.05 \end{bmatrix}$$

$$\mathbf{x}_2 = [0.300]$$

- Third mode of vibration

$$\mathbf{w}_3 = \begin{bmatrix} -0.002 \\ -0.250 \\ 1 \end{bmatrix}$$

$$\mathbf{x}_3 = [-0.355]$$
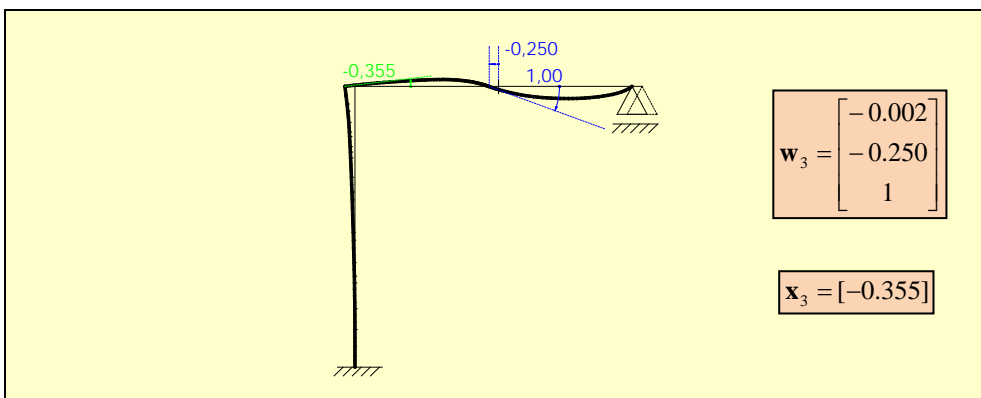
Harmonically forced vibration solution realized with the use of the Direct Method, Eqs. (3.95) and (3.96)

$$\begin{bmatrix} \mathbf{K} - p^2\mathbf{B} & \mathbf{0} \\ \mathbf{0} & \mathbf{K} - p^2\mathbf{B} \end{bmatrix} \cdot \begin{bmatrix} \mathbf{q}_S \\ \mathbf{q}_C \end{bmatrix} = \begin{bmatrix} \mathbf{F}_S \\ \mathbf{F}_C \end{bmatrix}$$
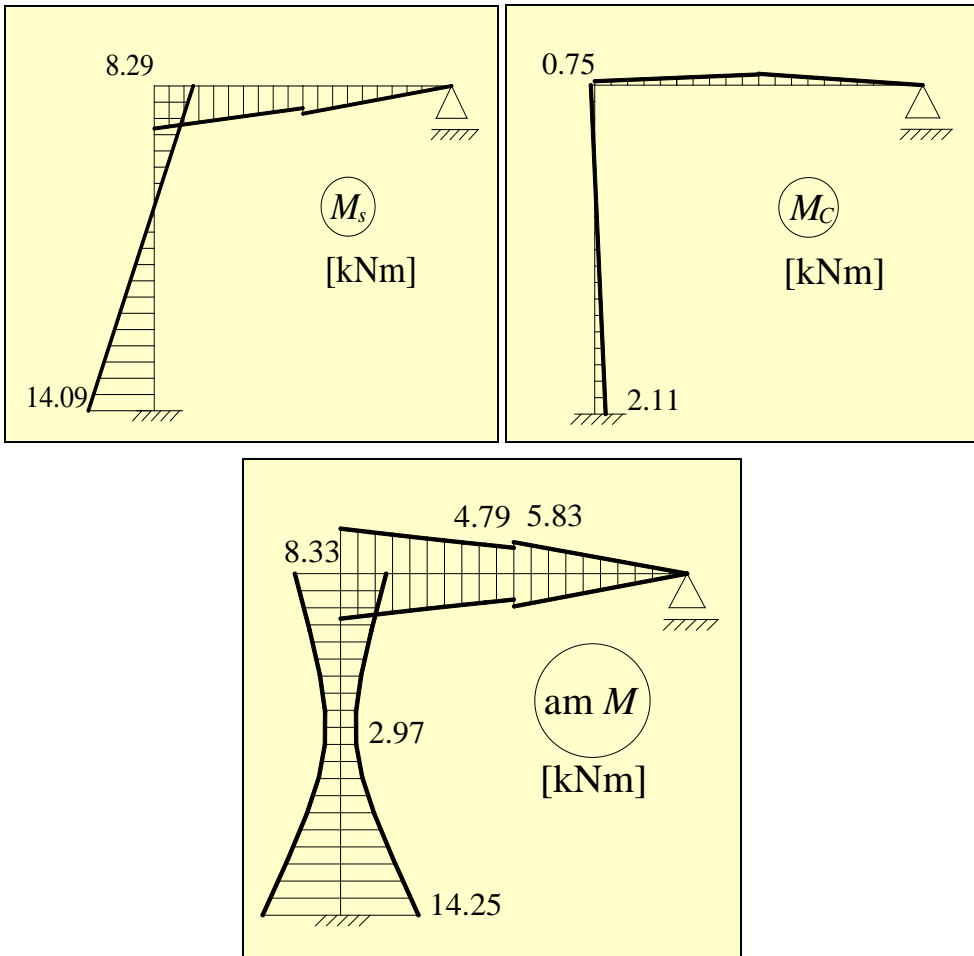
or

$$\mathbf{q}_{S,C} = (\mathbf{K} - p^2\mathbf{B})^{-1}\mathbf{F}_{S,C}$$

Finally, the displacements $\mathbf{q}_S$, $\mathbf{q}_C$ and am$\mathbf{q}$ are

$$\mathbf{q}_S = \begin{bmatrix} 0.001\,[\mathrm{m}] \\ 0.006\,[\mathrm{m}] \\ -7.8\cdot10^{-5}\,[\mathrm{rad}] \end{bmatrix}, \quad \mathbf{q}_C = \begin{bmatrix} -0.0003\,[\mathrm{m}] \\ -0.001\,[\mathrm{m}] \\ 6.6\cdot10^{-6}\,[\mathrm{rad}] \end{bmatrix}, \quad \mathrm{am}\,\mathbf{q} = \begin{bmatrix} 0.0011\,[\mathrm{m}] \\ 0.0061\,[\mathrm{m}] \\ 0.000078\,[\mathrm{rad}] \end{bmatrix},$$

The diagrams of bending moments M_S, M_C and am$M$ are shown in Figures below

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4.2. Trusses

Let us consider a statically determinate or indeterminate plane truss structure with accordance to the theory introduced by Langer in [5]. An example of a dynamic scheme of a truss structure is shown in Fig. 4.5.
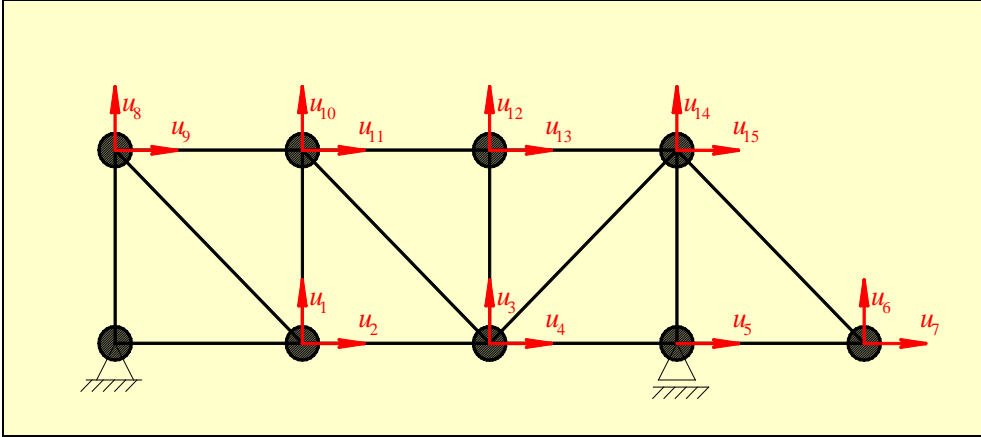

Fig. 4.5 Dynamic scheme of the truss

The prismatic and straight members are pin-connected. The length of a member is l_i, the area – A_i, the mass – m_i and Young's Modulus of Elasticity – E_i. It is assumed that the truss is a discrete structure whose masses m_i are concentrated in nodes. Each mass displacement is assumed to be described by two local cartesian coordinates. It is most convenient to assume that the generalized Lagrange's coordinates are the same as the local coordinates, i.e.

$$\mathbf{q} = \mathbf{u} \qquad \mathbf{A}_m = \mathbf{I} \qquad \mathbf{B} = \{\mathbf{m}\} \tag{4.33}$$

The mass matrix elements can be calculated from formula

$$m_i = M_i + \frac{1}{2}\sum_j m_j \tag{4.34}$$

where M_i is the concentrated mass associated with the generalized coordinate q_i and $\sum_j m_j$ is the sum of "j" mass of the members connected in node "i".

It is convenient to derive the equation of motion for the truss with the use of the Displacement Method, thus the stiffness matrix must be determined. The local coordinates of the displacement state are taken to be extensions of the members

$$\boldsymbol{\Delta} = [\Delta_1 \quad \Delta_2 \quad \Delta_3 \quad \ldots]^T \tag{4.35}$$

The transformation from generalized coordinates can be formulated as

$$\boldsymbol{\Delta} = \mathbf{A}_{\Delta} \mathbf{q}$$

(4.36)

The potential energy can be calculated as below

$$E_p = \frac{1}{2} \boldsymbol{\Delta}^T \left\{ \frac{\mathbf{EA}}{\mathbf{L}} \right\} \boldsymbol{\Delta} = \frac{1}{2} \mathbf{q}^T \mathbf{A}_{\Delta}^T \left\{ \frac{\mathbf{EA}}{\mathbf{L}} \right\} \mathbf{A}_{\Delta} \mathbf{q} = \frac{1}{2} \mathbf{q}^T \mathbf{K} \mathbf{q}$$

(4.37)

where

$$\mathbf{K} = \mathbf{A}_{\Delta}^T \left\{ \frac{\mathbf{EA}}{\mathbf{L}} \right\} \mathbf{A}_{\Delta}$$

(4.38)

The truss may be loaded by excitation forces acting on nodes of the structure. If these forces, represented by vector $\mathbf{P}$, are determined in the local base of coordinates $\mathbf{u}$, then, with regard to Eq. (4.33), $\mathbf{q} = \mathbf{u}$ and one can obtain $\mathbf{F} = \mathbf{P}$.

The damping matrix $\mathbf{C}$ ought to be determined with regard to the chosen damping model.

The equation of motion has the form Eq. (3.41). After the solution of this equation in the displacements base $\mathbf{q} = \mathbf{u}$, one ought to determine the stress-strain state in the form of the forces vector in the set of members

$$\mathbf{N} = \left\{ \frac{\mathbf{EA}}{\mathbf{L}} \right\} (\boldsymbol{\Delta} + \kappa \dot{\boldsymbol{\Delta}}) = \left\{ \frac{\mathbf{EA}}{\mathbf{L}} \right\} \mathbf{A}_{\Delta} (\mathbf{q} + \kappa \dot{\mathbf{q}})$$

(4.39)

In the case of the steady-state response to harmonic excitation, the damping matrix can be assumed to be in the form of Eq. (3.129) and then

$$\mathbf{N}_S = \left\{ \frac{\mathbf{EA}}{\mathbf{L}} \right\} \mathbf{A}_{\Delta} (\mathbf{q}_S - 2\alpha \mathbf{q}_C)$$

(4.40)

$$\mathbf{N}_C = \left\{ \frac{\mathbf{EA}}{\mathbf{L}} \right\} \mathbf{A}_{\Delta} (\mathbf{q}_C + 2\alpha \mathbf{q}_S)$$

(4.41)

$$\mathrm{am}N_j = \sqrt{N_{jS}^2 + N_{jC}^2}$$

(4.42)

where $\mathbf{q}_S$ and $\mathbf{q}_C$ fulfill Eq. (3.95).

If the damping influence is negligible, the state of displacement can be calculated from Eq. (3.96) and $\alpha = 0$ should be substituted into Eqs. (4.37) and (4.38).

4.3. Foundations

One of the most important problems in structural dynamics is the analysis of vibrations generated by a machine attached to a block foundation. The foundation is placed on the surface of elastic ground, Fig. 4.6.
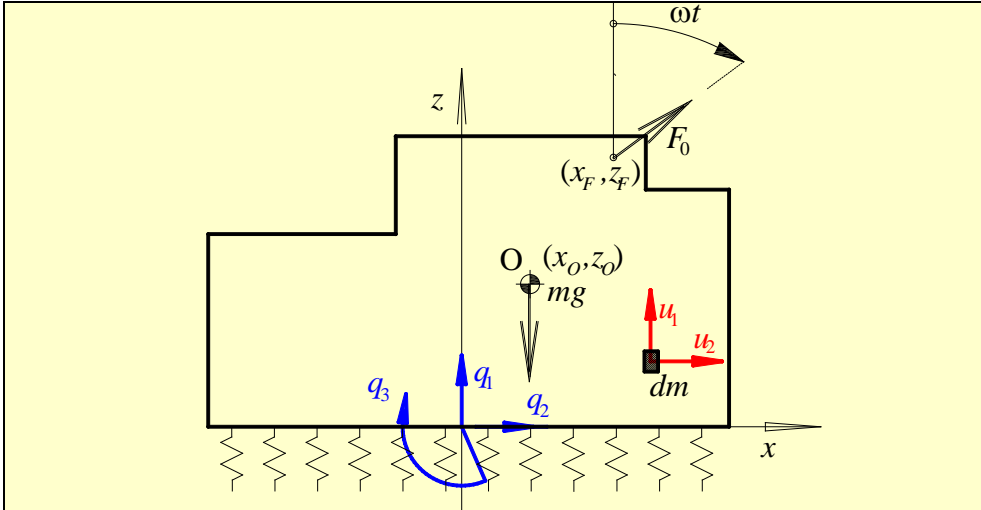

Fig. 4.6 Foundation on elastic ground

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Assumptions

It is assumed that:

- The block foundation is a rigid body.
- The base contact area is placed on the horizontal plane xy.
- The structure is symmetric with respect to the plane xz.
- The axes x, y are the principal axes of the area of the foundation contact surface.
- The axes x, y, z pass through the centroid of the area of the foundation contact surface.
- The elastic ground is a non-inertial one described by three parameters:
  - the stiffness coefficients in the horizontal direction are k_x, k_z (in the directions x, z respectively),
  - and the stiffness coefficient in the direction of rotation around the axis y perpendicular to the plane of vibrations is k_φ.
- The mass center of the block foundation (point O), could be located not on the same vertical line as the centroid of the foundation base area.
- The vibrations are harmonically excited by a force rotating in plane xz
- The force may be located not in the mass center of the block foundation.

</div>

Taking into account the above assumptions, the space vibration of the structure can be reduced to the problem of the plane vibrations in the symmetry plane xz. Three generalized coordinates are sufficient to describe the dynamic properties of the structure. These generalized, local and block mass center coordinates are shown in Fig. 4.6. In the vector notation they have a form

$$\mathbf{u} = [u_1 \quad u_2]^T \tag{4.43}$$

and

$$\mathbf{q} = [q_1 \quad q_2 \quad q_3]^T \tag{4.44}$$

The transformation of generalized to local coordinates can be written down in form

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -x \\ 0 & 1 & z \end{bmatrix} \cdot \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} \tag{4.45}$$

Finally, the inertia matrix has the form analogical to the one shown in Eq. (3.19), i.e.

$$\mathbf{B} = \begin{bmatrix} m & 0 & -S_{yz} \\ 0 & m & S_{xy} \\ -S_{yz} & S_{xy} & J_A \end{bmatrix} \tag{4.46}$$

where

$S_{xy} = m z_O$ The static moment of mass with respect to the plane xy.

$S_{yz} = m x_O$ The static moment of mass with respect to the plane yz.

$J_A = m(x_O^2 + z_O^2) + J_O$ The polar mass moment of inertia about the axis through the dynamic center – point A, i.e. about the axis y through the centroid of the contact area and perpendicular to the plane of vibrations.

J_O The moment of mass inertia of the machine-foundation structure with respect to the axis passing through the mass center of the machine-foundation structure

m The mass of machine-foundation structure

The potential energy of the ground elasticity is

$$E_p = \frac{1}{2}(k_z A q_1 + k_x A q_2 + k_\varphi J_A q_3) \tag{4.47}$$

From Eq. (4.47) it is possible to achieve the stiffness matrix

$$\mathbf{K} = \mathrm{diag}\left(k_z A \quad k_x A \quad k_\varphi J_A\right) \tag{4.48}$$

where A is the area of surface of contact between structure and elastic ground.

The damping matrix can be assumed on the basis of one of the hypotheses of damping, even though it is most frequently assumed that $\mathbf{C} = \kappa\mathbf{K}$, i.e. structural damping.

After reducing the rotating force from a given point of its localization to the point of localization of generalized coordinates. This point is the centroid A of the area of the foundation contact surface. The generalized forces vector can be obtained in the following form

$$\mathbf{F}(t) = \begin{bmatrix} 0 \\ 1 \\ z_F \end{bmatrix} \mathbf{F}_o \sin\omega t + \begin{bmatrix} 1 \\ 0 \\ -x_F \end{bmatrix} \mathbf{F}_o \cos\omega t \qquad (4.49)$$

Since the weight of the block foundation is usually significant, and because of the assumption that the mass center, point O, of the block foundation is not located on the same vertical line as the centroid of the foundation base area, it seems to be advisable to take into account an amendment resulting from the second order theory. This second order theory amendment is, in this situation, the moment of the gravity force (weight of the block foundation) about the axis y through the centroid of the foundation base area. This axis is perpendicular to the plane of vibrations. Since the mass center, point O, of the block foundation is moving during the vibrations, this additional moment of force depends on time. This additional moment of force can be written down in the form

$$\Delta\mathbf{F}(t) = \begin{bmatrix} 0 \\ 0 \\ mg\, z_O \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & g\, S_{xy} \end{bmatrix} \cdot \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} \qquad (4.50)$$

This vector should be transferred onto the left side of the matrix equation of motion. After this operation the matrix equation of motion has the standard form

$$\mathbf{B}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{F}(t) \qquad (4.51)$$

The stiffness matrix is diagonal but differs from the form presented in Eq. (4.48). Now,

$$\mathbf{K} = \mathrm{diag}\,(\, k_z\, A \qquad k_x\, A \qquad k_\varphi\, J_A - g\, S_{xy}\,) \qquad (4.52)$$

The solution of Eq. (4.51) can be achieved with the use of the Direct Method or the Modal Transformation Method. As a result the vector $\mathbf{q}$ is obtained and then it is possible to calculate the vector $\mathbf{Q}$ of the generalized forces transmitted to the ground

$$\mathbf{Q} = \mathbf{F}_T = \mathbf{F} + \omega^2\mathbf{B}\mathbf{q} \qquad (4.53)$$

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There are three elements in this vector. The first element $Q_1 = F_{Tz}$ is the vertical transmitted force; the second element $Q_2 = F_{Tx}$ is the horizontal transmitted force; the third element $Q_3 = M_{Ty}$ is the moment of transmitted force around the axis y. These transmitted forces make it possible to find the dynamic stresses in the foundation contact surface. These stresses ought to be calculated separately for sinusoidal and cosinusoidal components. The final amplitudal values of stresses should be calculated with the use of formula

$$\mathrm{am}\,\sigma = \sqrt{\sigma_S^2 + \sigma_C^2}$$
(4.45)

The generalized coordinates vector is also useful to obtain local displacements of chosen points of foundation, with the use of formula Eq. (4.46). The amplitude of the local cartesian displacements of any point may be obtained from expressions

$$\mathrm{am}\,u_1 = \sqrt{u_{1S}^2 + u_{1C}^2}$$
(4.55)

and

$$\mathrm{am}\,u_2 = \sqrt{u_{2S}^2 + u_{2C}^2}$$
(4.56)

These amplitudes describe the "frame of trajectory" of the vibrating point. As it was described in Chapter 1.4.2, the trajectory is an ellipse, see Fig. 4.7 and Fig. 1.5 (Lissajous figures – periodic vibration). This ellipse is inscribed into a rectangular frame with dimensions $(2\mathrm{am}\,u_1) \times (2\mathrm{am}\,u_2)$.
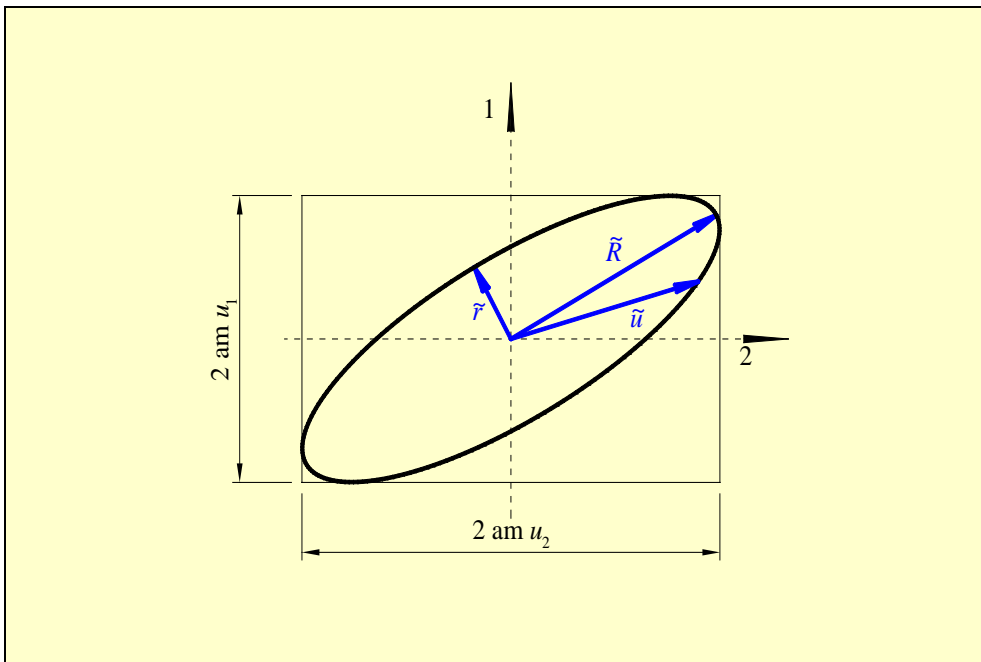

Fig. 4.7 Trajectory of vibration of a given point *A* of block foundation.

It is important to notice that the dimensions of the "frame of trajectory" change depending on the directions of the local cartesian coordinates, while the trajectory itself is an invariant figure, i.e. it describes the motion of the point in an objective way. For this reason it is important to determine the trajectory. It can be found that

$$u^2(t) = \mathbf{u}^T\mathbf{u} = (\mathbf{u}_S^T \sin \omega t + \mathbf{u}_C^T \cos \omega t)\cdot(\mathbf{u}_S \sin \omega t + \mathbf{u}_C \cos \omega t) =$$
$$= \frac{1}{2}(\mathbf{u}_C^T\mathbf{u}_C + \mathbf{u}_S^T\mathbf{u}_S) + \sqrt{\frac{1}{4}(\mathbf{u}_C^T\mathbf{u}_C - \mathbf{u}_S^T\mathbf{u}_S)^2 + (\mathbf{u}_S^T\mathbf{u}_C)^2}\,\cos(2\omega t - 2\theta)$$
(4.57)

where

$$2\theta = \arctan\frac{2\mathbf{u}_S^T\mathbf{u}_C}{\mathbf{u}_C^T\mathbf{u}_C - \mathbf{u}_S^T\mathbf{u}_S}$$
(4.58)

and

$$\left.\begin{array}{l}\tilde{R} = \max u(t)\\ \tilde{r} = \min u(t)\end{array}\right\} = \sqrt{\frac{1}{2}(\mathbf{u}_C^T\mathbf{u}_C + \mathbf{u}_S^T\mathbf{u}_S) \pm \sqrt{\frac{1}{4}(\mathbf{u}_C^T\mathbf{u}_C - \mathbf{u}_S^T\mathbf{u}_S)^2 + (\mathbf{u}_S^T\mathbf{u}_C)^2}}$$
(4.59)

The formulas in Eq. (4.60) describe the values of the principal amplitudes, but they do not describe their geometric orientations. The maximum value of the principle amplitude occurs when $\cos(2\omega t - 2\theta) = 1$, which means that $\omega t = \theta$. Then

$$\tilde{\mathbf{R}} = \begin{bmatrix}\tilde{R}_1\\ \tilde{R}_2\end{bmatrix} = \mathbf{u}_S \sin\theta + \mathbf{u}_C \cos\theta$$
(4.60)

The minimum value of the principle amplitude occurs when $\cos(2\omega t - 2\theta) = -1$, which means that $\omega t = \theta + \pi/2$. Then

$$\tilde{\mathbf{r}} = \begin{bmatrix}\tilde{r}_1\\ \tilde{r}_2\end{bmatrix} = \mathbf{u}_S \sin\theta + \mathbf{u}_C \cos\theta$$
(4.61)

Expressions Eqs. (4.61), (4.62) describe the principle amplitudes in the vector sense by their projections on coordinate axes x and z. Therefore, it is possible to determine the geometrical orientation of the motion trajectory. Moreover, it is also possible to use the relations

$$\tilde{R} = \sqrt{\tilde{\mathbf{R}}^T\tilde{\mathbf{R}}}$$
(4.62)

and

$$\tilde{r} = \sqrt{\tilde{\mathbf{r}}^T\tilde{\mathbf{r}}}$$
(4.63)

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