

DIDACTICS OF MATHEMATICS

10(14)



The Publishing House
of Wrocław University of Economics
Wrocław 2013

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Information of submitting and reviewing paper is available on
the Publishing House's website www.wydawnictwo.ue.wroc.pl

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Wrocław 2013

PL ISSN 1733-7941

The original version: printed

Printing: Printing House EXPOL,
P. Rybiński, J. Dąbek, sp.j. ul. Brzeska 4, 87-800 Włocławek

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**SOME REMARKS ON HORIZONTAL,
SLANT, PARABOLIC
AND POLYNOMIAL ASYMPTOTE****Tadeusz Janaszak**

Abstract. In the teaching of calculus, we consider horizontal and slant asymptote. In this paper the idea of asymptote of function is expanded to polynomials. There are given formulas of coefficients of the multinomial asymptote of the function and some examples of parabolic and cubic asymptote.

Keywords: Horizontal asymptote, slant asymptote, parabolic asymptote, cubic asymptote, multinomial asymptote, rational function.

DOI: 10.15611/dm.2013.10.04

1. Horizontal and slant asymptote

The term ‘asymptote’ means usually a straight line, thus a line l is an asymptote to a curve if the distance from point P to the line l tends to zero as P tends to infinity along some unbounded part of the curve. If the curve is the graph of a real function this definition includes a vertical $x = x_0$, **a horizontal** $y = a_0$, and **a slant asymptote** $y = a_1 \cdot x + a_0$ (Clapham 1996; Kudravcev 1973).

It is known that the horizontal asymptote of function $y = f(x)$ has its parameter $a_0 = \lim_{x \rightarrow \infty} f(x)$, if it is investigated as $x \rightarrow \infty$, the slant asymptote has its parameters:

$$a_1 = \lim_{x \rightarrow \infty} \frac{f(x)}{x} \quad \text{and} \quad a_0 = \lim_{x \rightarrow \infty} (f(x) - a_1 \cdot x).$$

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In the case $x \rightarrow -\infty$ the result can be different. Function $y = \arctan x$ has two different asymptotes if

$$x \rightarrow -\infty \left(y = -\frac{\pi}{2} \right) \text{ and } y = \frac{\pi}{2} \text{ if } x \rightarrow \infty.$$

The same is the case for slant, parabolic and polynomial asymptote.

The reader certainly knows many examples of the function with horizontal or slant asymptote.

Example 1. The most simple example of a function which has the horizontal asymptote $y = 0$ is $y = \frac{1}{x}$.

Example 2. The sum: $f(x) = \text{const} + 1/x$ has the horizontal asymptote $y = \text{const}$.

Example 3. The sum of the linear function $y = ax + b$ and the hyperbola from example 1 is a rational function and its slant asymptote is the linear function: $y = x + 2 + \frac{1}{x}$:

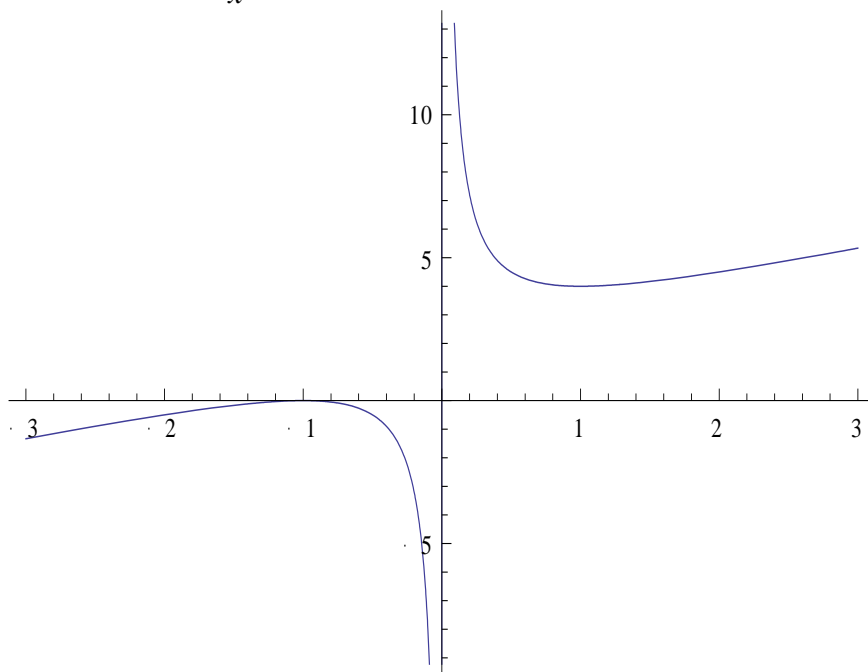


Fig. 1. A rational function with the slant asymptote

Source: own elaboration.

Example 4. Function $f(x) = \frac{\sin x}{x}$ is not rational and has the horizontal asymptote $y = 0$:

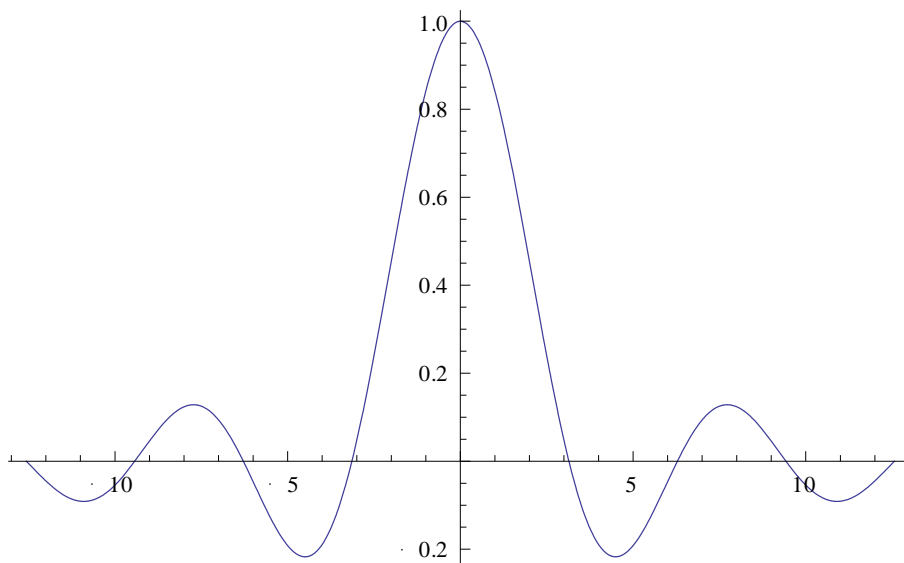


Fig. 2. The function oscillates closer and closer around its horizontal asymptote
Source: own elaboration.

Example 5. Let $f(x) = ax + b + \frac{\sin x}{x}$ where $a \neq 0$. This function oscillates closer and closer around its slant asymptote $y = ax + b$.

2. Parabolic asymptote

The term ‘parabolic asymptote’ is defined similarly (Janaszak 2000).

Definition 1. A parabola $y = a_2 \cdot x^2 + a_1 \cdot x + a_0$ is a **parabolic asymptote** of the function $y = f(x)$, as $x \rightarrow \infty$, if

$$\lim_{x \rightarrow \infty} \left(f(x) - (a_2 x^2 + a_1 x + a_0) \right) = 0. \quad (1)$$

In the case $x \rightarrow -\infty$ the result can be different.

Theorem 1. Let function f be given in any interval (m, ∞) , and have the parabolic asymptote defined by (1), then there exist three limits:

$$\lim_{x \rightarrow \infty} \frac{f(x)}{x^2} = a, \quad (2)$$

$$\lim_{x \rightarrow \infty} \frac{f(x) - a_2 \cdot x^2}{x} = b, \quad (3)$$

$$\lim_{x \rightarrow \infty} f(x) - (a_2 \cdot x^2 + a_1 \cdot x) = c, \quad (4)$$

and the parameters a_2, a_1, a_0 are equal: a, b, c .

Proof. The equalities (1) and (4) are equivalent as $c = a_0$.

Let ε be a positive number. By formula (1) there exists a number $M_1 > \frac{|a_0|}{\varepsilon}$ such that for each $x > M_1$ inequality

$$|f(x) - (a_2 \cdot x^2 + a_1 \cdot x) - a_0| < \varepsilon_1 \quad (5)$$

where $\varepsilon_1 = M_1 \cdot \varepsilon - |a_0| > 0$, holds. The inequality (5) can be divided by $x > M_1$, then the sequence of inequalities

$$-\frac{|a_0| + \varepsilon_1}{M_1} \leq -\frac{|a_0| + \varepsilon_1}{x} < \frac{f(x) - (a_2 \cdot x^2 + a_1 \cdot x)}{x} < \frac{|a_0| + \varepsilon_1}{x} \leq \frac{|a_0| + \varepsilon_1}{M_1}$$

holds, hence

$$-\varepsilon < \frac{f(x) - (a_2 \cdot x^2 + a_1 \cdot x)}{x} < \varepsilon,$$

the inequality above proves that

$$\lim_{x \rightarrow \infty} \frac{f(x) - (a_2 \cdot x^2 + a_1 \cdot x)}{x} = 0 \quad (6)$$

i.e.

$$\lim_{x \rightarrow \infty} \frac{f(x) - a_2 \cdot x^2}{x} = a_1, \quad (7)$$

thus the limit (3) exists, and $a_1 = b$.

Now we need to prove that the equality (2) holds, and $a_2 = c$. Let $\varepsilon > 0$, by (7) there exists a number $M_2 > \frac{|a_1|}{\varepsilon}$ such that for each $x > M_2$ double inequality

$$a_1 - \varepsilon_2 < \frac{f(x) - a_2 \cdot x^2}{x} < a_1 + \varepsilon_2$$

holds, where $\varepsilon_2 = M_2 \cdot \varepsilon - a_1 > 0$. Because

$$a_1 + \varepsilon_2 \leq |a_1| + \varepsilon_2 \quad \text{and} \quad -|a_1| - \varepsilon_2 \leq a_1 - \varepsilon_2,$$

the inequality below is true:

$$-|a_1| - \varepsilon_2 < \frac{f(x) - a_2 \cdot x^2}{x} < |a_1| + \varepsilon_2.$$

Now the above inequality is divided by $x > M_2$:

$$-\frac{|a_1| + \varepsilon_2}{M_2} < -\frac{|a_1| + \varepsilon_2}{x} < \frac{f(x) - a_2 \cdot x^2}{x^2} < \frac{|a_1| + \varepsilon_2}{x} < \frac{|a_1| + \varepsilon_2}{M_2}.$$

We have

$$-\varepsilon < \frac{f(x) - a_2 \cdot x^2}{x^2} < \varepsilon. \tag{8}$$

The double inequality (8) implies the equality (2) where $a = a_2$. The proof is complete.

Corollary 1. The function $y = f(x)$ has the parabolic asymptote, as $x \rightarrow \infty$, if and only if the formulas (2), (3), (4) hold; the parabola is given by formula $y = ax^2 + bx + c$.

Corollary 2. Function f has at most one parabolic asymptote.

The proof can also be made directly by definition. If the formula (1) holds and the equality

$$\lim_{x \rightarrow \infty} (c_2 x^2 + c_1 x + c_0 - f(x)) = 0 \tag{9}$$

holds too, then the addition of the left side of (1) and (9) has the limit equal to zero, i.e.

$$\lim_{x \rightarrow \infty} \left((c_2x^2 + c_1x + c_0) - (a_2x^2 + a_1x + a_0) \right) = 0, \quad (10)$$

so

$$\lim_{x \rightarrow \infty} x^2 \cdot \left((c_2 - a_2) + \frac{c_1 - a_1}{x} + \frac{c_0 - a_0}{x^2} \right) = 0; \quad (11)$$

the formula (11) proves that $c_2 = a_2$, from here it follows that the formula (10) is equivalent to

$$\lim_{x \rightarrow \infty} \left((c_1x + c_0) - (a_1x + a_0) \right) = 0 \quad (12)$$

i.e.

$$\lim_{x \rightarrow \infty} x \cdot \left((c_1 - a_1) + \frac{c_0 - a_0}{x} \right) = 0; \quad (13)$$

the formula (13) entails that $c_1 = a_1$, and the formula (10) is equivalent to $\lim_{x \rightarrow \infty} (c_0 - a_0) = 0$ i.e. $c_0 = a_0$. The direct proof of corollary 2 is complete.

Example 6. Let a rational function f be given by the following formula:

$$f(x) = \frac{x^3 - 6x^2 + 11x - 6}{x}.$$

Its decomposition has the form:

$$f(x) = x^2 - 6x + 11 - \frac{6}{x}.$$

The parabolic asymptote of f is defined by $y = x^2 - 6x + 11$ with the vertex in the point $x = 3$, $y = 2$. Function f has the form:

$$f(x) = \frac{(x-1) \cdot (x-2) \cdot (x-3)}{x}.$$

The table of signs of f helps to made the plot of the function f ; it runs above the x -axis on the intervals

$$(-\infty, 0); (1, 2); (3, \infty)$$

and below it on the intervals

$$(0, 1) \text{ and } (2, 3).$$

Similarly, it is located above its parabolic asymptote on the interval $(-\infty, 0)$, and below on $(0, \infty)$. In Figure 3 there is presented a graph of function f , and its parabolic asymptote.

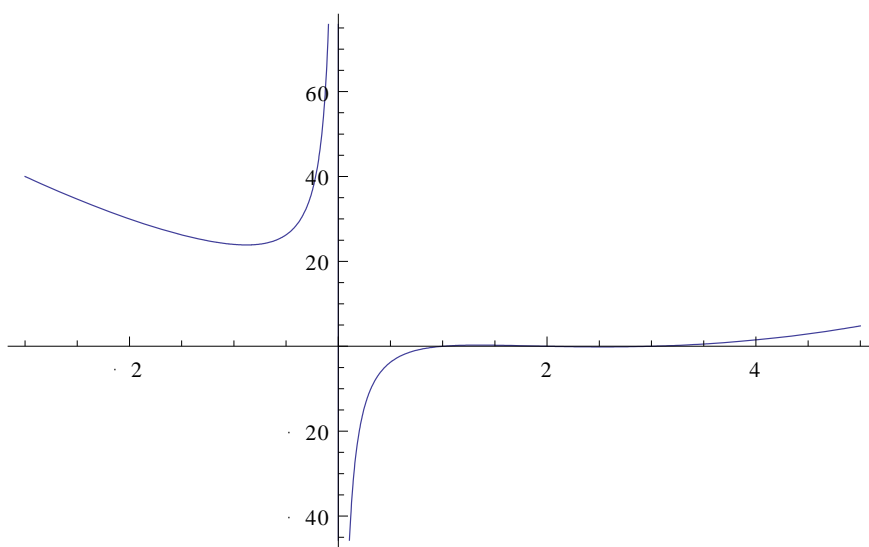


Fig. 3. A rational function with its parabolic asymptote

Source: own elaboration.

Example 7. The function given by the formula

$$g(x) = x^2 - 6x + 11 + \frac{\sin x}{x}$$

has the same parabolic asymptote as f in example 6; as $x \rightarrow \pm\infty$. The graph of it is a sinusoid which runs closer and closer around the parabola $y = x^2 - 6x + 11$.

3. Polynomial asymptote

Similarly to parabolic asymptote, there is defined the term ‘polynomial asymptote’.

Definition 2. A polynomial

$$p(x) = \sum_{i=0}^n a_i x^i \tag{14}$$

is said to be a **polynomial or multinomial asymptote** of the function $y = f(x)$, as $x \rightarrow \infty$, if the equality

$$\lim_{x \rightarrow \infty} (f(x) - p(x)) = 0 \quad (15)$$

holds. In the case $x \rightarrow -\infty$ the result can be different.

Theorem 2. Let a polynomial $p(x)$ given by (14) be a multinomial asymptote of function $f(x)$ and the domain of function f includes any interval (m, ∞) . Then there exists $n + 1$ limits:

$$\lim_{x \rightarrow \infty} \frac{f(x)}{x^n} = b_n, \quad (16)$$

$$\lim_{x \rightarrow \infty} \frac{f(x) - a_n x^n}{x^{n-1}} = b_{n-1}, \quad (17)$$

and so on

$$\lim_{x \rightarrow \infty} \frac{f(x) - \sum_{i=k+1}^n a_i x^i}{x^k} = b_k, \quad (18)$$

and so on

$$\lim_{x \rightarrow \infty} \frac{f(x) - \sum_{i=2}^n a_i x^i}{x} = b_1, \quad (19)$$

$$\lim_{x \rightarrow \infty} \frac{f(x) - \sum_{i=1}^n a_i x^i}{1} = b_0, \quad (20)$$

and for each $i = 0, \dots, n$ the equality $a_i = b_i$ holds.

Proof. The proof is made by induction. The first induction step is trivial because the equalities (15) and (20) are equivalent when $b_0 = a_0$.

Second induction step: let the formula (18) be true for each $k = 0, 1, \dots, r$, and $a_i = b_i$ for each $i = 0, \dots, r$, where r is a natural number: such that $0 \leq r \leq n - 2$. It needs to be proved that

$$\lim_{x \rightarrow \infty} \frac{f(x) - \sum_{i=k+2}^n a_i x^i}{x^{r+1}} = a_{r+1}. \quad (21)$$

By the induction assumption the formula

$$\lim_{x \rightarrow \infty} \frac{f(x) - \sum_{i=r+1}^n a_i x^i}{x^r} = a_r \quad (22)$$

holds. Let $\varepsilon > 0$, by (22) there exists $M_{r+1} > \frac{|a_{k+1}|}{\varepsilon}$ such that for each $x > M_{r+1}$ the double inequality below is true:

$$a_r - \varepsilon_{r+1} < \frac{f(x) - \sum_{i=r+1}^n a_i x^i}{x^r} < a_r + \varepsilon_{r+1},$$

where $\varepsilon_{r+1} = M_{r+1} \cdot \varepsilon - a_r > 0$. The inequality above can be broadened:

$$|a_r| - \varepsilon_{r+1} \leq a_r - \varepsilon_{r+1} < \frac{f(x) - \sum_{i=r+2}^n a_i x^i - a_{r+1} x^{r+1}}{x^r} < a_r + \varepsilon_{r+1} \leq |a_r| + \varepsilon_{r+1}.$$

The last inequality is divided now by $x > M_{r+1}$:

$$\begin{aligned} a_{r+1} - \frac{|a_r| + \varepsilon_{r+1}}{M_{r+1}} &< a_{r+1} - \frac{|a_r| + \varepsilon_{r+1}}{x} < \frac{f(x) - \sum_{i=r+2}^n a_i x^i}{x^{r+1}} \\ &< a_{r+1} + \frac{|a_r| + \varepsilon_{r+1}}{x} < a_{r+1} + \frac{|a_r| + \varepsilon_{r+1}}{M_{r+1}} \end{aligned}$$

that is

$$a_{r+1} - \varepsilon < \frac{f(x) - \sum_{i=r+2}^n a_i x^i}{x^{r+1}} < a_{r+1} + \varepsilon;$$

the inequality above proves that

$$\lim_{x \rightarrow \infty} \frac{f(x) - \sum_{i=k+1}^n a_i x^i}{x^k} = a_k \quad (23)$$

for each $k = 0, \dots, n-1$. It is necessary to prove that

$$\lim_{x \rightarrow \infty} \frac{f(x)}{x^n} = a_n. \quad (24)$$

Let it now be noted

$$\lim_{x \rightarrow \infty} \frac{f(x) - a_n x^n}{x^{n-1}} = a_{n-1}. \quad (25)$$

For each $\varepsilon > 0$ there exists $M_n > \frac{|a_{n-1}|}{\varepsilon}$ such that for each $x > M_n$, and for $\varepsilon_n = M_n \cdot \varepsilon - a_{n-1}$ the inequality

$$\left| \frac{f(x) - a_n \cdot x^n - a_{n-1} \cdot x^{n-1}}{x^{n-1}} \right| < \varepsilon_n \quad (26)$$

holds. Hence the inequality below holds too:

$$\left| \frac{f(x) - a_n \cdot x^n}{x^{n-1}} \right| < |a_{n-1}| + \varepsilon_n. \quad (27)$$

For the conclusion of the proof, we need to divide (27) by $x > M_n$:

$$\left| \frac{f(x) - a_n \cdot x^n}{x^n} \right| < \frac{|a_{n-1}| + \varepsilon_n}{x} < \frac{|a_{n-1}| + \varepsilon_n}{M_n} = \varepsilon \quad (28)$$

and so the equality (24) holds.

Corollary 3. Function f has at the most one polynomial asymptote. The coefficients of the polynomial are given by the formulas (16)-(20).

Theorem 3. Suppose the function $y = f(x)$ has the polynomial asymptote $y = p(x)$ which is given by (14). Then f can be represented by the sum

$$f(x) = p(x) + r(x) \quad (29)$$

where

$$\lim_{x \rightarrow \infty} r(x) = 0. \quad (30)$$

The polynomial p is said to be the principal part of function f , with respect to the set of multinomials, and r the remainder part of it.

Proof. By assumption the equality (15) holds, function r is given by the formula

$$r(x) = f(x) - p(x) \quad (31)$$

from here

$$p(x) + r(x) = p(x) + (f(x) - p(x)) = f(x); \quad (32)$$

the proof is complete.

Theorem 4. Let $f(x)$ be a rational function:

$$f(x) = \frac{P(x)}{Q(x)} \quad (33)$$

where $P(x)$ and $Q(x)$ are polynomials. Then f has a decomposition:

$$f(x) = p(x) + \frac{P_1(x)}{Q(x)} \quad (34)$$

where p and P_1 are polynomials too, and the degree of P_1 is less than Q . The polynomial p is the principal or integer part of the rational function f and $\frac{P_1}{Q}$ is the remainder or fractional part of it.

Example 8. The function

$$f(x) = \frac{x^5 - 5x^3 + 4x + 3}{x^2 - 1}$$

is rational and has the decomposition

$$f(x) = x^3 - 4x + \frac{3}{x^2 - 1}.$$

The polynomial $y = x^3 - 4x$ is the multinomial asymptote of the function. It is not difficult to make a graph of the function and its asymptote.

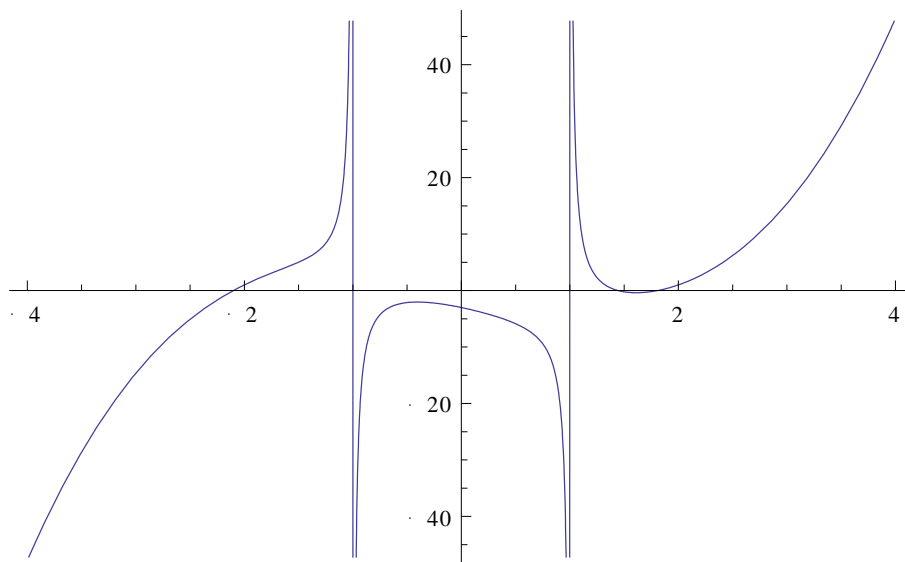


Fig. 4. A rational function with a cubic asymptote

Source: own elaboration.

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