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LOWER-LIMIT BARRIER IN THE PROBLEM OF THE IDENTIFICATION OF A BARRIER IN THE FUNCTIONING OF A CERTAIN INVENTORY STORAGE AND ISSUE SYSTEM

The paper investigates a certain inventory system whose input is a non-aggregated dynamic-parameter process. The authors derive equations that define the distribution of conditional probabilities for the case of a lower-limit barier in subsystem *L*. They depend on the parameters of the functioning of transport subsystem and the parameters of the process of product supply to finite-volume storage.

Keywords: *system, barrier, inventory*

1. Introduction

Paper [1] defined the notion of a barrier in the functioning of an inventory storage and issue system with non-aggregated dynamic-parameter input and presented the general operating principles of such a system and the conditional distributions for intermediate states of the subsystem *L*. These distributions were used in paper [2] to derive relations satisfied by the density functions for intermediate states.

Continuing the investigations reported in [1], [2], this paper contains derivation of the analytical forms of the conditional probabilities in the case of a lower-limit barrier in a process controlled by a dynamic-parameter input.

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2. Conditional probabilities in the case of a lower-limit barrier in the subsystem *L*

The lower-limit barrier of the subsystem *L* at time $t \in T_l$ is defined by the random event $z(t) = 0$. The state can be characterized by probabilities of the form:

$$Q_k^{ul}(\{0\},t) = P(z(t) = 0,\ x(t) = x_k,\ v(t) = u),\ t \in T_l,\ u = 1,0. \tag{1}$$

These probabilities satisfy the following relations:
for $u = 0$,

$$\begin{aligned} Q_k^{0l}(\{0\},t+\tau) &= P(z(t+\tau) = 0, x(t+\tau) = x_k, v(t+\tau) = 0) = \\ &= \sum_i \int_0^{V_1} q_{ik}^{00l}(z,\{0\};\tau,t) Q_i^{0l}(dz,t) + \int_0^{V_1} q_{kk}^{10l}(z,\{0\};\tau,t) Q_k^{1l}(dz,t), \end{aligned} \tag{2}$$

where

$$\begin{aligned} q_{ik}^{00l}(z,\{0\};\tau,t) = P(x(t) \text{ has one jump for } t+\theta \in (t,t+\tau), v(s) = 0 \text{ for } s \in [t,t+\tau], \\ z - d\tau \le 0 \,|\, x(t) = x_i,\ v(t) = 0) + o_1^{(l)}(\tau); \\ i \ne k, z = z(t) \in [0,V_1],\ t \in T_l,\ t+\tau \in T_l, \end{aligned} \tag{3}$$

$$\begin{aligned} q_{kk}^{00l}(z,\{0\};\tau,t) = P(x(s) = x_k, s \in [t,t+\tau], v(s) = 0 \text{ for } s \in [t,t+\tau], \\ z - d\tau \le 0 \,|\, x(t) = x_k, v(t) = 0) + o_1^{(l)}(\tau); \\ z = z(t) \in [0,V_1],\ t \in T_l,\ t+\tau \in T_l, \end{aligned} \tag{4}$$

$$\begin{aligned} q_{kk}^{10l}(z,\{0\};\tau,t) = P(x(s) = x_k, \text{ for } s \in [t,t+\tau], v(t) \text{ has one jump for } t+\theta \in (t,t+\tau), \\ h(z + x_k\theta) - d(\tau - \theta) \le 0 \,|\, x(t) = x_k, v(t) = 1) + o_1^{(l)}(\tau); \\ z = z(t) \in [0,V_1],\ t \in T_l,\ t+\tau \in T_l, \end{aligned} \tag{5}$$

for $u = 1$,

$$\begin{aligned} Q_k^{1l}(\{0\},t+\tau) &= P(z(t+\tau) = 0,\ x(t+\tau) = x_k,\ v(t+\tau) = 1) = \\ &= \sum_i \int_0^{V_1} q_{ik}^{11l}(z,\{0\};\tau,t) Q_i^{1l}(dz,t) + \int_0^{V_1} q_{kk}^{01l}(z,\{0\};\tau,t) Q_k^{0l}(dz,t), \end{aligned} \tag{6}$$

where

$$\begin{aligned} q_{ik}^{11l}(z,\{0\};\tau,t) = P(x(t) \text{ has one jump for } t+\theta \in (t,t+\tau),\ v(s) = 1 \text{ for } s \in [t,t+\tau], \\ h(z + \theta x_i) - (\tau - \theta)x_k \le 0 \,|\, x(t) = x_i, v(t) = 1) + o_1^{(l)}(\tau); \\ i \ne k, z = z(t) \in [0,V_1],\ t \in T_l,\ t+\tau \in T_l, \end{aligned} \tag{7}$$

$$q_{kk}^{11l}(z,\{0\};\tau,t) = P(x(s) = x_k, v(s) = 1 \text{ for } s \in [t, t+\tau], \\ z + \tau x_k \le 0 \,\big|\, x(t) = x_k, v(t) = 1) + o_1^{(l)}(\tau); \\ z = z(t) \in [0, V_1],\ t \in T_l,\ t+\tau \in T_l, \tag{8}$$

$$q_{kk}^{01l}(z,\{0\};\tau,t) = P(x(s) = x_k \text{ for } s \in [t, t+\tau], v(t) \text{ has one jump for } t+\theta \in (t, t+\tau), \\ h(z - d\theta) + x_k(\tau - \theta) \le 0 \,\big|\, x(t) = x_k, v(t) = 0) + o_1^{(l)}(\tau); \\ z = z(t) \in [0, V_1],\ t \in T_l,\ t+\tau \in T_l. \tag{9}$$

In order to find the probabilities $Q_k^{1l}(\{0\};t+\tau)$ and $Q_k^{0l}(\{0\};t+\tau)$, we have to find the conditional probabilities $q_{ik}^{11l}(z,\{0\};\tau,t)$, $q_{kk}^{11l}(z,\{0\};\tau,t)$, $q_{kk}^{01l}(z,\{0\};\tau,t)$, $q_{ik}^{00l}\left(z,\{0\};\tau,t\right)$, $q_{kk}^{10l}(z,\{0\};\tau,t)$, $q_{kk}^{00l}(z,\{0\};\tau,t)$. Taking into account relations (3) and (6) from [1], we obtain

$$q_{ik}^{11l}(z,\{0\};\tau,t) = \exp(-\pi_1^{*l}\tau)\int_B \frac{\pi_{ik}^{(l)}}{\pi_i^{(l)}} d(1 - \exp(-\pi_i^{(l)}\theta)) + o_1^{(l)}(\tau), \tag{10}$$

where

$$B = \{\theta{:}0 < \theta < \tau;\ h(z + \theta x_i) + (\tau - \theta)x_k \le 0\}, \\ i \ne k,\ z = z(t) \in [0, V_1],\ t \in T_l,\ t+\tau \in T_l. \tag{11}$$

If $x_i > 0,\ x_k < 0,\ \tau$ small, $0 < \theta < \tau,\ t \in T_l,\ t+\tau \in T_l,$ then

$$\{\theta{:}\, h(z + \theta x_i) + (\tau - \theta)x_k \le 0\} = \{\theta{:}\, z + \theta x_i + (\tau - \theta)x_k \le 0\} \cap \{\theta{:}\, z + \theta x_i < V_1\} =$$
$$= \begin{cases} \left\{\theta{:}\, 0 \le \theta \le \dfrac{-z - \tau x_k}{x_i - x_k}\right\} \cap \left\{\theta{:}\, \theta < \dfrac{V_1 - z}{x_i}\right\}, & z < -\tau x_k, \\ \varnothing, & \text{other } z. \end{cases}$$

Thus

$$B = \begin{cases} \left\{\theta{:}0 < \theta \le \dfrac{z + \tau x_k}{x_k - x_i}\right\}, & 0 \le z < -\tau x_k, \\ \varnothing, & \text{other } z. \end{cases}$$

Hence, from (10), by integration, we get

$$q_{ik}^{11l}(z,\{0\};\tau,t) - o_1^{(l)}(\tau) = \begin{cases} (1 - \pi_1^{*l}\tau)\pi_{ik}^{(l)} \dfrac{z + \tau x_k}{x_k - x_i} + o^{(l)}(\tau; z), & 0 \le z < -\tau x_k, \\ 0, & \text{other } z. \end{cases} \tag{12}$$

Similarly it can be shown that the conditional probabilities are given by the following formulas:

for $x_k > 0,\ x_i$ – any state,

$$q_{ik}^{11l}(z,\{0\};\tau,t) = o_1^{(l)}(\tau),\ 0 \le z \le V_1, \tag{13}$$

for any $x_k,\ x_i$,

$$q_{ik}^{00l}(z,\{0\};\tau,t) - o_1^{(l)}(\tau) = \begin{cases} (1-\pi_0^{*l}\tau)\pi_{ik}^{(l)}\tau + o^{(l)}(\tau), & 0 \le z < d\tau, \\ 0, & \text{other } z, \end{cases} \tag{14}$$

$$q_{kk}^{00l}(z,\{0\};\tau,t) - o_1^{(l)}(\tau) = \begin{cases} (1-\pi_0^{*l}\tau)(1-\pi_k^{(l)}\tau) + o^{(l)}(\tau), & 0 \le z < d\tau, \\ 0, & \text{other } z, \end{cases} \tag{15}$$

for $x_i < 0,\ x_k = 0$,

$$q_{ik}^{11l}(z,\{0\};\tau,t) - o_1^{(l)}(\tau) = \begin{cases} (1-\pi_1^{*l}\tau)\pi_{ik}^{(l)}\left(z + \dfrac{z}{x_i}\right) + o^{(l)}(\tau;z), & 0 \le z < -\tau x_i, \\ 0, & \text{other } z, \end{cases} \tag{16}$$

for $x_i < 0,\ x_k < 0$,

$$q_{ik}^{11l}(z,\{0\};\tau,t) - o_1^{(l)}(\tau) =$$

$$= \begin{cases} (1-\pi_1^{*l}\tau)\pi_{ik}^{(l)}\tau + o^{(l)}(\tau), & x_i > x_k, 0 \le z < -\tau x_i, \\ (1-\pi_1^{*l}\tau)\pi_{ik}^{(l)}\dfrac{z+\tau x_k}{x_k - x_i} + o^{(l)}(\tau;z), & x_i > x_k, -\tau x_i < z < -\tau x_k, \\ 0, & x_i > x_k,\ \text{other } z, \\ (1-\pi_1^{*l}\tau)\pi_{ik}^{(l)}\tau + o^{(l)}(\tau), & x_i < x_k, 0 \le z < -\tau x_k, \\ (1-\pi_1^{*l}\tau)\pi_{ik}^{(l)}\left(\tau - \dfrac{z+\tau x_k}{x_k - x_i}\right) + o^{(l)}(\tau;z), & x_i < x_k, -\tau x_k < z < -\tau x_i, \\ 0, & x_i < x_k,\ \text{other } z, \end{cases} \tag{17}$$

for $x_k \le 0$,

$$q_{kk}^{11l}(z,\{0\};\tau,t) - o_1^{(l)}(\tau) = \begin{cases} (1-\pi_k^{(l)}\tau)(1-\pi_1^{*l}\tau) + o^{(l)}(\tau), & 0 \le z < \tau x_k, \\ 0, & \text{other } z, \end{cases} \tag{18}$$

for $x_k > 0$,

$$q_{kk}^{01l}(z,\{0\};\tau,t) - o_1^{(l)}(\tau) = 0,\ 0 \le z \le V_1, \tag{19}$$

for $x_k = 0$,

$$q_{kk}^{01l}(z,\{0\};\tau,t) - o_1^{(l)}(\tau) = \begin{cases} (1-\pi_k^{(l)}\tau)\,\pi_0^{*l}\left(\tau - \dfrac{z}{d}\right) + o^{(l)}(\tau;z), & 0 \le z < \tau\, d, \\ 0, & \text{other } z, \end{cases} \tag{20}$$

for $x_k < 0$,

$$q_{kk}^{01l}(z,\{0\};\tau,t) - o_1^{(l)}(\tau) =$$

$$= \begin{cases} (1-\pi_k^{(l)}\tau)\pi_0^{*l}\tau + o^{(l)}(\tau), & d + x_k \le 0, 0 \le z < d\tau, \\ 0, & d + x_k \le 0, \text{ other } z, \\ (1-\pi_k^{(l)}\tau)\pi_0^{*l}\tau + o^{(l)}(\tau), & d + x_k > 0, 0 \le z < -x_k\tau, \\ (1-\pi_k^{(l)}\tau)\pi_0^{*l}\left(\tau - \dfrac{z+\tau x_k}{d+x_k}\right) + o^{(l)}(\tau;z), & d + x_k > 0, -x_k\tau < z < d\tau, \\ 0, & d + x_k > 0, \text{ other } z, \end{cases} \tag{21}$$

for $x_k < 0$,

$$q_{kk}^{10l}(z,\{0\};\tau,t) - o_1^{(l)}(\tau) =$$

$$= \begin{cases} (1-\pi_k^{(l)}\tau)\pi_1^{*l}\tau + o^{(l)}(\tau), & d + x_k \le 0, 0 \le z < d\tau, \\ 0, & d + x_k \le 0, \text{ other } z, \\ (1-\pi_k^{(l)}\tau)\pi_1^{*l}\tau + o^{(l)}(\tau), & d + x_k > 0, 0 \le z < -x_k\tau, \\ (1-\pi_k^{(l)}\tau)\pi_1^{*l}\dfrac{d\tau - z}{d+x_k} + o^{(l)}(\tau;z), & d + x_k > 0, -x_k\tau < z < d\tau, \\ 0, & d + x_k > 0, \text{ other } z, \end{cases} \tag{22}$$

for $x_k > 0$,

$$q_{kk}^{10l}(z,\{0\};\tau,t) - o_1^{(l)}(\tau) = \begin{cases} (1-\pi_k^{(l)}\tau)\,\pi_1^{*l}\,\dfrac{d\tau - z}{d+x_k} + o^{(l)}(\tau;z), & 0 \le z < d\tau, \\ 0, & \text{other } z, \end{cases} \tag{23}$$

for $x_k = 0$,

$$q_{kk}^{10l}(z,\{0\};\tau,t) - o_1^{(l)}(\tau) = \begin{cases} (1-\pi_k^{(l)}\tau)\,\pi_1^{*l}\,\dfrac{d\tau - z}{d} + o^{(l)}(\tau;z), & 0 \le z < d\tau, \\ 0, & \text{other } z, \end{cases} \tag{24}$$

for $x_k > 0$,

$$q_{kk}^{11l}(z,\{0\};\tau,t) - o_1^{(l)}(\tau) = 0,\ 0 \le z \le V_1. \tag{25}$$

The components $o^{(l)}(\tau; ...)$, $o_1^{(l)}(\tau)$, which appear in equations (12)–(25) are in the vicinity of $\tau = 0$ infinitely small quantities of an order higher than τ. The relations (12)–(25) will be used in the authors' next paper to derive relations satisfied by the probabilities $Q_k^{ul}(\{0\};t+\tau)$. These will be used for quantitative identification of a barrier in the functioning of the system under consideration.

References

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Bariera dolna procesu w zagadnieniu identyfikacji i bariery funkcjonowania pewnego systemu gromadzenia i wydawania zapasów

Badana jest bariera działania pewnego systemu gospodarki zapasami, którego wejście jest procesem niezagregowanym o dynamicznych parametrach. Wyprowadzono wzory wyrażające warunkowe rozkłady prawdopodobieństwa w przypadku bariery dolnej podsystemu *L*. Zależą one od parametrów funkcjonowania podsystemu transportowego oraz parametrów procesu podaży produktu do magazynu o skończonej objętości.

Słowa kluczowe: *system, bariera, zapas*