M A T H E M A T I C A L   E C O N O M I C S

No. 12(19) 2016

VORTICES ON A PINCHED SPHERE

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Abstract: The vortex motion is a sign of the desire to achieve balance. In the study, vortices described by complex numbers are moved from a sphere into a "squeezed" sphere – where the pair of opposite poles become one – called a pinched sphere or concave disk. Vortices on the pinched sphere reflect what is commonly observed in nature. A family of the pinched spheres very well represents the spatial vortices observed daily in gusts of wind. Stock market zigzags constitute an economic vortex – a spiral on the cone whose equivalent is the pinched sphere.

Keywords: pinched sphere – concave disk, vortex, complex number.

JEL Classification: C13, C65.

DOI: 10.15611/me.2016.12.05.

1. Prelude

In many works (see for example [Alobaidi et al. 2006; Boatto, Koiller 2015; Eto et al. 2006; Knio, Ghoniem 1990]) vortices are described with differential equations on differentiable manifolds, such as a sphere, cylinder or torus. By identifying the vortex with a complex number, one obtains a simple model of the basic phenomena of nature [Maciuk, Smoluk 2015]. In the work the vortices defined on the sphere are naturally moved to the pinched sphere.

2. Concave disk

A family of rotating cones $z^2 = c^2(x^2 + y^2)$ wherein $c \in \mathbb{R}_+^*$ then $c > 0$ is associated with the sphere $z^2 = r^2 - x^2 - y^2$, $r > 0$. As a result of this operation one obtains a surface family, due to its shape – called pinched spheres or concave disks,

$$z^2 = c^2(x^2 + y^2)(r^2 - x^2 - y^2).$$

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If c grows at infinity then pinched spheres mutually approach to the circle $x^2 + y^2 \leq r^2$. If c decreases to zero, then pinched spheres within the border break down into the straight $x = y = 0$ and the circle $x^2 + y^2 = r^2$. A compactified cone is a homeomorphic surface with the sphere $x^2 + y^2 + z^2 = r^2$, in which the south pole $A = (0,0,-r)$ was equated with the north $B = (0,0,r)$. This cone is a homeomorphic surface with the pinched sphere. In this compactification and identification, straight lines creating (Figure 1) are transferred to a line that is topologically equivalent to two tangential circles (Figure 2). A pinched sphere can be formed by splitting a sphere along the equator, the connection of the north and south pole, and then folding the edges on the outside and attaching them along the equator.

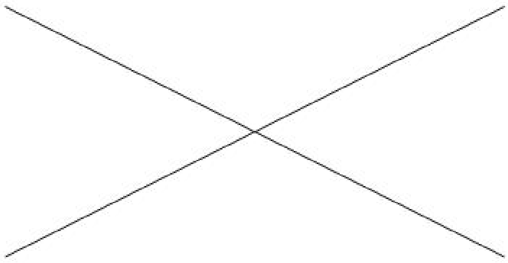

Fig. 1. Cross section of a cone

Source: own elaboration.

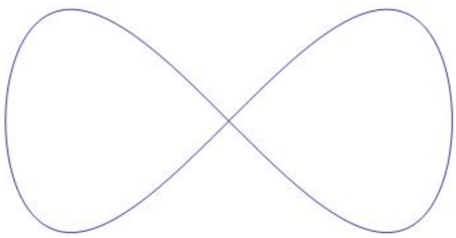

Fig. 2. Compactification

Source: own elaboration.

Parametric equations of pinched sphere A_c data in the formulas:

$$\begin{cases} x = \rho \cos \alpha \\ y = \rho \sin \alpha \\ z = c\rho\sqrt{(r^2 - \rho^2)} \end{cases},$$

where $0 \leq \alpha \leq 2\pi$ and $-r \leq \rho \leq r$. A pinched sphere is a crater just like a Greek vase or a volcano pit, where the parameter c determines its height. It is a smooth manifold, compact, edgeless, with one singular point $(0,0,0)$ – a spike. For large values of the parameter c the surface is similar to that connected to two high flute-type glasses forming a symmetrical arrangement. For small c the figure is a flattened horizontal eight, or two flat plates connected. The upper surface reaches the maximum value $M(c) = \frac{2}{3\sqrt{3}}cr^3$ in points lying on a circle $x^2 + y^2 = \frac{r^2}{2}$. In the point (0,0) it is the minimum of 0. The point is, as mentioned above, the attractor for the top surface and a repulsor for the bottom surface. The lower surface has on the circle $x^2 + y^2 = \frac{r^2}{2}$ a minimum $m(c)$, and in the point (0,0) a maximum. Of course $m(c) = -M(c)$.

A cross-section of pinched sphere is a plane passing through the axis z at an angle specified by the parameter $\alpha = 0$ is the line in Figure 2 (∞). This line consists of two functions: the top one, having maxima at the points $x_1 = \frac{r}{\sqrt{2}}$, $x_2 = -\frac{r}{\sqrt{2}}$ and a minimum at $x = 0$, and lower function, symmetrical to the upper one and having two minimums and one maximum.

Let us enrich the pinched spheres family with a new parameter that allows flexible surface modeling.

The two-parameter family of surface:

$$z = \pm c(x^2 + y^2)^k\sqrt{r^2 - x^2 - y^2},$$

where $k > 0$ and $c > 0$, is still called a family of pinched spheres. The case of $k = 1$ was considered above. If $k = 2$, then the surface of the pinched sphere is a smooth manifold also with a singularity at (0,0,0). At this point there is one contact surface – there is no spike. The top portion reaches the maximum surface $M_2(c) = \frac{2}{3\sqrt{3}}cr^3$ in points on the circle $x^2 + y^2 = \frac{2}{3}r^2$ and a minimum value of 0 in point $(0,0)$. Accordingly, the extremes change on the bottom surface. Figure 5 shows cross-sections of pinched spheres when

$c = 1$ and k equals respectively ¼, ½, 1, 2, and 4. For large values of k small c pinched spheres resemble symmetrically stacked plates, for large k and c symmetrical stacked mugs $c = 1$.

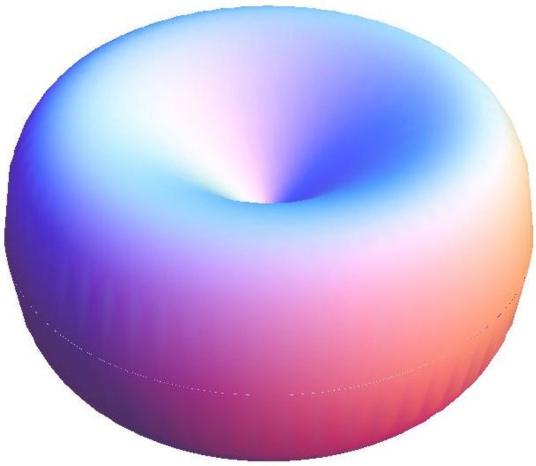

Fig. 3. Pinched sphere

Source: own elaboration.

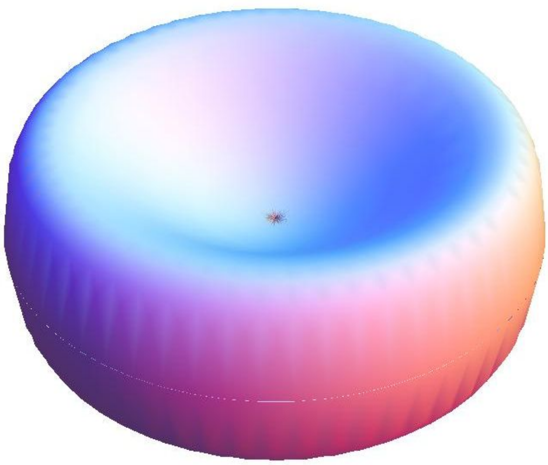

Fig. 4. Pinched sphere for $k = 2$

Source: own elaboration.

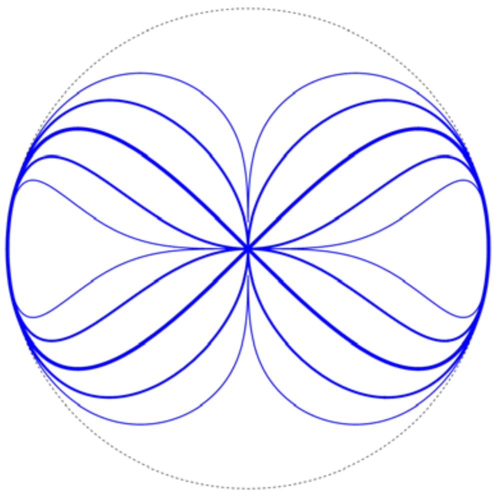

Fig. 5. Selection of the pinched spheres' cross-sections
for c = 1 and k respectively ¼, ½, 1, 2, 4

Source: own elaboration.

3. Vortex on a concave disk

A flat vortex from primary sphere $x^2 + y^2 + z^2 = r^2$, hence the two-dimensional surface which it identifies with a plane compactified with one point, has its equivalent in each pinched sphere

$$A_c, z^2 = c^2(x^2 + y^2)(r^2 - x^2 - y^2),$$

where $c > 0$. As the result a spatial vortex is generated by one complex number [Maciuk, Smoluk 2015]. The base parametric equation for the spiral formed on pinched spheres:

$$f_{\rho,\phi,\psi,k}(t) =$$

$$\left(\rho^{-|t|}\cos(\phi t + \psi), \rho^{-|t|}\sin(\phi t + \psi), \rho^{-k|t|}\text{sign}(t)\sqrt{1 - \rho^{-2|t|}}\right),$$

where ρ is the modulus of the complex number $A = \rho e^{i\phi} = a + bi$, ϕ – the angle generated by this number, ψ – the angle related to the number $u \in \mathbb{T}$, $k > 0$ – parameter modeling the shape of the surface of the pinched sphere and $t \in \mathbb{R}$.

A vortex is an attribute of equilibrium: either it indicates a pursuit of the state of equilibrium or, conversely, an increase of chaos – drawing far from equilibrium. An empirical vortex, which is observed on the stock exchange or in the dance of leaves in the wind, we describe as a complex number. This complex number can be determined e.g. using the method of least squares. Each complex number generates a spiral trajectory family. The least squares method helps to choose the trajectory that best matches the observed empirical data. This procedure is analogous to smoothing the data by the least square method using an appropriate function family. Here, as the equivalent of a family function we have spiral lines that are generated by complex numbers.

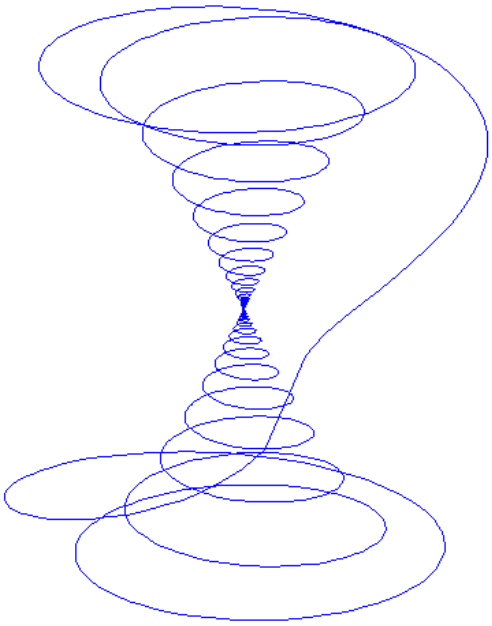

Fig. 6. An example of a vortex on the surface of a pinched sphere

Source: own elaboration.

If a trajectory extends on the upper plane of the pinched sphere to the point (0,0,0) then the system stabilizes itself – it seeks equilibrium. Point (0,0,0) in this case is an attractor. If we are at the bottom surface and the trajectory turns away from of the point (0,0,0), the system destabilizes – it tends to disaster. Point (0,0,0) in this case is repulse. Of course, the point (0,0,0) will in practice be a model, a description of the real object. In the case of the stock exchange, this is the equivalent to the apex of the cone in the space defined by time, share price and supply volume generated by the zig-zags showing the changes of the rates. Another example of the vortex is a widely observed water vortex flowing out of a tub or the vortices formed on a river. The result of those water vortices are known from geology as plunge pools, i.e. limestone crevices, hollows in the rock in the shape of a pinched sphere.

4. Finale

Torus T^2 is a smooth manifold, which we all know from experience; a bicycle tube and a bagel are torus-shaped. This manifold is a topological two-dimensional group as a product of a wheel group $T = \{e^{it}: t \in \mathbb{R}\}$. On the torus one can observe three types of vortices: small vortices with the equation $(e^{it}, e^{\beta i})$, where $t \in \mathbb{R}$ and $0 \leq \beta \leq 2\pi$ is a predetermined angle, a large vortex – which is a rotation in relation to a large wheel - of equation $\left(e^{\alpha i}, e^{ti}\right)$ wherein a $t \in \mathbb{R}$ and $0 \leq \alpha \leq 2\pi$. At the end we have a spiral vortex that is a combination of the previous two vortices and consists of orbital movement with equation $(e^{it}, e^{f(t)i})$ where the function $f: \mathbb{R} \to \mathbb{R}$. This third type of vortices, characteristic for the torus, is observed in the course of subtropical winds – trade winds. One could say that a vortex motion rules the world, because the world is moving into balance, and the vortex is a sign of this trend.

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