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CAN TORI ARISE IN A TWO-REGIONAL MODEL WITH FIXED EXCHANGE RATES?***

A two-regional, five dimensional model describing the development of income, capital stock and money stock, which was introduced by T. Asada in [1], is analysed. Sufficient conditions are found for the existence of two pairs of purely imaginary eigenvalues and a fifth negative one for the linear approximation matrix of the model. A theorem on the existence of invariant tori is presented.

Keywords: *dynamic model, equilibrium, linear approximatiomatrix, eigenvalues, normal form of differential equations on invariant surface, bifurcation equation, torus*

1. Introduction

In [2] T. Asada, T. Inaba and T. Misawa developed and studied a two regional model of business cycles with fixed exchange rates, which consists of a five dimensional discrete time system. In [1] T. Asada introduced and analysed a continuous time version of this model, describing the dynamic interaction of two regions which are connected through interregional trade and capital movement. Throughout the paper we adopt the notation used in [1] with minor changes. The analysed model is of the form

$$\begin{gathered} \dot{Y}_i = \alpha_i(C_i + I_i + G_i + J_i - Y_i), \alpha_i > 0, \\ \dot{K}_i = I_i, \\ \dot{M}_1 = p_1 A_1, \quad i = 1,2, \end{gathered} \tag{1}$$

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where:

$$C_i = c_i(Y_i - T_i) + C_{0i},\ 0 < c_i < 1, C_{0i} \geq 0,$$

$$T_i = \tau_i Y_i - T_{0i},\ 0 < \tau_i < 1,\ T_{0i} \geq 0,$$

$$I_i = I_i(Y_i, K_i, r_i),\ \frac{\partial I_i}{\partial Y_i} > 0,\ \frac{\partial I_i}{\partial K_i} < 0,\ \frac{\partial I_i}{\partial r_i} < 0,$$

$$\frac{M_i}{p_i} = L_i(Y_i, r_i),\ \frac{\partial L_i}{\partial Y_i} > 0,\ \frac{\partial L_i}{\partial r_i} < 0,$$

$$J_1 = J_1(Y_1, Y_2, E),\ \frac{\partial J_1}{\partial Y_1} < 0,\ \frac{\partial J_1}{\partial Y_2} > 0,\ \frac{\partial J_1}{\partial E} > 0, \tag{2}$$

$$Q_1 = \beta\left(r_1 - r_2 - \frac{E^e - E}{E}\right),\ \beta > 0,$$

$$A_1 = J_1 + Q_1,$$

$$0 = p_1 J_1 + E p_2 J_2,$$

$$0 = p_1 Q_1 + E p_2 Q_2,$$

$$\overline{M} = M_1 + E M_2,$$

and the subscript i, $i = 1, 2$, is the index of a region. The meanings of the symbols in (1) and (2) are as follows: Y_i – real regional income, K_i – real physical capital stock, M_i – nominal money stock, L_i – demand for money, C_i – consumption, T_i – taxes, I_i – net real private investment expenditure on physical capital, G_i - real government expenditure (fixed), p_i – price level, r_i – nominal rate of interest, E – exchange rate, E^e – expected exchange rate, J_i – current account balance (net exports) in real terms, Q_i – capital account in real terms (net capital inflow), A_i – total balance of payments in real terms, α_i – adjustment speed in goods market and β – degree of capital mobility.

As Asada in [1], in this paper we assume fixed price economy with fixed exchange rates. Therefore, normalizing the price levels in the two regions, we can suppose that $p_1 = p_2 = 1$, $E = E^e$. Furthermore we suppose that the nominal interest rate r_i, $i = 1, 2$, adjusts instantaneously to keep the money stock M_i and demand for money L_i in equilibrium. Under these assumptions, taking into account (2) and supposing that r_i is implicitly determined by the relations $M_i = L_i(Y_i, r_i),\ i = 1, 2$, model (1) takes the form

$$\dot{Y}_1 = \alpha_1[c_1(1-\tau_1)Y_1 + c_1 T_{01} + C_{01} + G_1 + I_1(Y_1, K_1, r_1(Y_1, M_1)) + J_1(Y_1, Y_2, E) - Y_1],$$

$$\dot{K}_1 = I_1(Y_1, K_1, r_1(Y_1, M_1)),$$

$$\dot{Y}_2 = \alpha_2 \Bigg[c_2(1-\tau_2)Y_2 + c_2T_{02} + C_{02} + G_2 \qquad (3)$$

$$+ I_2\left(Y_2, K_2, r_2\left(Y_2, \frac{\overline{M}-M_1}{E}\right)\right) - \frac{1}{E}J_1(Y_1,Y_2,E) - Y_2 \Bigg],$$

$$\dot{K}_2 = I_2\left(Y_2, K_2, r_2\left(Y_2, \frac{\overline{M}-M_1}{E}\right)\right),$$

$$\dot{M}_1 = J_1(Y_1,Y_2,E) + \beta\left[r_1(Y_1,M_1) - r_2\left(Y_2, \frac{\overline{M}-M_1}{E}\right)\right].$$

In the whole article we suppose that:

1. Model (3) has a unique equilibrium point $(Y_{10}, K_{10}, Y_{20}, K_{20}, M_{10})$ with positive coordinates for an arbitrary triple of positive parameters $(\alpha_1, \alpha_2, \beta)$.

2. All functions in model (3) are linear with respect to their variables, except the functions I_i and r_i, $i = 1,2$, which are nonlinear in Y_i of type C^6 in a small neighbourhood of the equilibrium point.

Remark 1. The analysis of the existence of an equilibrium for model (3) was performed by Asada in [1]. The requirement on the functions in (3) to be of class C^6 with respect to Y_i enables the transformation of model (3) to its partial normal form on an invariant surface and to use a theorem on the existence of tori (see e.g. [3]).

In [1] Asada found sufficient conditions for local stability of the equilibrium point. The question of the existence of business cycles around the equilibrium is analysed in [1]. In the present paper we are interested in the existence of tori in a small neighbourhood of the equilibrium point. Tori can arise only in the case when the linear approximation matrix of model (3) at the equilibrium point has two pairs of purely imaginary eigenvalues. Model (3) is analysed in section 2. Theorem 1 gives sufficient conditions for the existence of two pairs of purely imaginary eigenvalues with the remaining one being negative. Theorem 2 comments on the existence of tori in a small neighbourhood of the equilibrium point.

2. The analysis of model (3)

Let us write model (3) in the abbreviated form

$$\dot{\xi} = \Xi(\xi; \alpha_1, \alpha_2, \beta),$$

where $\xi = (Y_1, K_1, Y_2, K_2, M_1)$. After translating the equilibrium point $\xi_0 = (Y_{10}, K_{10}, Y_{20}, K_{20}, M_{10})$ to the origin via the coordinate shift $y = \xi - \xi_0$, model (3) becomes

$$\dot{y} = \Xi(y + \xi_0; \alpha_1, \alpha_2, \beta).$$

Its Taylor expansion at $y = 0$ gives

$$\dot{y} = A(\alpha_1, \alpha_2, \beta)y + Y(y; \alpha_1, \alpha_2, \beta), \tag{4}$$

where $Y(y; \alpha_1, \alpha_2, \beta) = O(\| y \|^2)$ and the linear approximation matrix $A(\alpha_1, \alpha_2, \beta)$ is

$$A(\alpha_1, \alpha_2, \beta) = \begin{pmatrix} \alpha_1 G_{11} & \alpha_1 G_{12} & \alpha_1 G_{13} & 0 & \alpha_1 G_{15} \\ F_{21} & G_{12} & 0 & 0 & G_{15} \\ \alpha_2 G_{31} & 0 & \alpha_2 G_{33} & \alpha_2 G_{34} & \alpha_2 G_{35} \\ 0 & 0 & F_{43} & G_{34} & G_{35} \\ F_{51}(\beta) & 0 & F_{53}(\beta) & 0 & F_{55}(\beta) \end{pmatrix}, \tag{5}$$

where:

$$G_{11} = \frac{\partial I_1}{\partial Y_1} + \frac{\partial I_1}{\partial r_1}\frac{\partial r_1}{\partial Y_1} - \left[1 - c_1(1-\tau_1) - \frac{\partial J_1}{\partial Y_1}\right],$$

$$G_{12} = \frac{\partial I_1}{\partial K_1} < 0,$$

$$G_{13} = \frac{\partial J_1}{\partial Y_2} > 0,$$

$$G_{15} = \frac{\partial I_1}{\partial r_1}\frac{\partial r_1}{\partial M_1} > 0,$$

$$F_{21} = \frac{\partial I_1}{\partial Y_1} + \frac{\partial I_1}{\partial r_1}\frac{\partial r_1}{\partial Y_1},$$

$$G_{31} = -\frac{1}{E}\frac{\partial J_1}{\partial Y_1} > 0,$$

$$G_{33} = \frac{\partial I_2}{\partial Y_2} + \frac{\partial I_2}{\partial r_2}\frac{\partial r_2}{\partial Y_2} - \left[1 - c_2(1-\tau_2) + \frac{1}{E}\frac{\partial J_1}{\partial Y_2}\right],$$

$$G_{34} = \frac{\partial I_2}{\partial K_2} < 0,$$

$$G_{35} = -\frac{1}{E}\frac{\partial I_2}{\partial r_2}\frac{\partial r_2}{\partial M_2} < 0,$$

$$F_{43} = \frac{\partial I_2}{\partial Y_2} + \frac{\partial I_2}{\partial r_2}\frac{\partial r_2}{\partial Y_2},$$

$$F_{51}(\beta) = \frac{\partial J_1}{\partial Y_1} + \beta\frac{\partial r_1}{\partial Y_1},$$

$$F_{53}(\beta) = \frac{\partial J_1}{\partial Y_2} - \beta\frac{\partial r_2}{\partial Y_2},$$

$$F_{55}(\beta) = \beta\left[\frac{\partial r_1}{\partial M_1} + \frac{1}{E}\frac{\partial r_2}{\partial M_2}\right] < 0,$$

while all the derivatives are evaluated at the equilibrium point ξ_0.

Assumption 1. As in [1], we asume that the values of $\frac{\partial I_1(\xi_0)}{\partial Y_1}$ and $\frac{\partial I_2(\xi_0)}{\partial Y_2}$ are sufficiently large that $G_{11} > 0$ and $G_{33} > 0$ at the equilibrium point ξ_0.

Assumption 2. The value of the parameter β can be considered arbitrarily large compared with the values of the parameters α_1 and α_2.

Remark 2. Assumption 1 is analogous to the standard hypothesis of the Kaldorian model of business cycles [4]. This assumption automatically implies that $F_{21} > 0$ and $F_{43} > 0$ Assumption 2 also agrees with economic theory.

Definition. A triple $(\alpha_{10}, \alpha_{20}, \beta_0)$ of parameters α_1, α_2 and β in (4) is called a critical triple of model (4) if the matrix $A(\alpha_{10}, \alpha_{20}, \beta_0)$ has eigenvalues $\lambda_{1,2} = \pm i\omega_1$, $\lambda_{3,4} = \pm i\omega_2$, $i = \sqrt{-1}$, $\lambda_5 < 0$.

The characteristic equation of $A(\alpha_1, \alpha_2, \beta)$ is

$$\lambda^5 + a_1\lambda^4 + a_2\lambda^4 + a_3\lambda^2 + a_4\lambda + a_5 = 0, \tag{6}$$

where:

$a_1 = a_1(\alpha_1, \alpha_2, \beta) = -\text{trace } A = -\alpha_1 G_{11} - G_{12} - \alpha_2 G_{33} - G_{34} - F_{55}(\beta),$

$a_2 = a_2(\alpha_1, \alpha_2, \beta) =$ sum of all the principal second-order minors of A,

$$a_2 = \alpha_1 \begin{vmatrix} G_{11} & G_{12} \\ F_{21} & G_{12} \end{vmatrix} + \alpha_1\alpha_2 \begin{vmatrix} G_{11} & G_{13} \\ G_{31} & G_{33} \end{vmatrix} + \alpha_1 \begin{vmatrix} G_{11} & 0 \\ 0 & G_{34} \end{vmatrix}$$

$$+ \alpha_1 \begin{vmatrix} G_{11} & G_{15} \\ F_{51}(\beta) & F_{55}(\beta) \end{vmatrix} + \alpha_2 \begin{vmatrix} G_{12} & 0 \\ 0 & G_{33} \end{vmatrix} + \begin{vmatrix} G_{12} & 0 \\ 0 & G_{34} \end{vmatrix}$$

$$+ \begin{vmatrix} G_{12} & G_{15} \\ 0 & F_{55}(\beta) \end{vmatrix} + \alpha_2 \begin{vmatrix} G_{33} & G_{34} \\ F_{43} & G_{34} \end{vmatrix} + \alpha_2 \begin{vmatrix} G_{33} & G_{35} \\ F_{53}(\beta) & F_{55}(\beta) \end{vmatrix} + \begin{vmatrix} G_{34} & G_{35} \\ 0 & F_{55}(\beta) \end{vmatrix},$$

$a_3 = a_3(\alpha_1, \alpha_2, \beta) = -$ sum of all the principal third-order minors of A,

$$a_3 = -\alpha_1\alpha_2 \begin{vmatrix} G_{11} & G_{12} & G_{13} \\ F_{21} & G_{12} & 0 \\ G_{31} & 0 & G_{33} \end{vmatrix} - \alpha_1 \begin{vmatrix} G_{11} & G_{12} & 0 \\ F_{21} & G_{12} & 0 \\ 0 & 0 & G_{34} \end{vmatrix}$$

$$- \alpha_1 \begin{vmatrix} G_{11} & G_{12} & G_{15} \\ F_{21} & G_{12} & G_{15} \\ F_{51}(\beta) & 0 & F_{55}(\beta) \end{vmatrix} - \alpha_1\alpha_2 \begin{vmatrix} G_{11} & G_{13} & 0 \\ G_{31} & G_{33} & G_{34} \\ 0 & F_{43} & G_{34} \end{vmatrix}$$

$$- \alpha_1\alpha_2 \begin{vmatrix} G_{11} & G_{13} & G_{15} \\ G_{31} & G_{33} & G_{35} \\ F_{51}(\beta) & F_{53}(\beta) & F_{55}(\beta) \end{vmatrix} - \alpha_1 \begin{vmatrix} G_{11} & 0 & G_{15} \\ 0 & G_{34} & G_{35} \\ F_{51}(\beta) & 0 & F_{55}(\beta) \end{vmatrix}$$

$$- \alpha_2 \begin{vmatrix} G_{12} & 0 & 0 \\ 0 & G_{33} & G_{34} \\ 0 & F_{43} & G_{34} \end{vmatrix} - \alpha_2 \begin{vmatrix} G_{12} & 0 & G_{15} \\ 0 & G_{33} & G_{35} \\ 0 & F_{53}(\beta) & F_{55}(\beta) \end{vmatrix}$$

$$- \begin{vmatrix} G_{12} & 0 & G_{15} \\ 0 & G_{34} & G_{35} \\ 0 & 0 & F_{55}(\beta) \end{vmatrix} - \alpha_2 \begin{vmatrix} G_{33} & G_{34} & G_{35} \\ F_{43} & G_{34} & G_{35} \\ F_{53}(\beta) & 0 & F_{55}(\beta) \end{vmatrix},$$

$a_4 = a_4(\alpha_1, \alpha_2, \beta) =$ sum of all the principal fourth-order minors of A,

$$a_4 = \alpha_1\alpha_2 \begin{vmatrix} G_{11} & G_{12} & G_{13} & 0 \\ F_{21} & G_{12} & 0 & 0 \\ G_{31} & 0 & G_{33} & G_{34} \\ 0 & 0 & F_{43} & G_{34} \end{vmatrix} + \alpha_1\alpha_2 \begin{vmatrix} G_{11} & G_{12} & G_{13} & G_{15} \\ F_{21} & G_{12} & 0 & G_{15} \\ G_{31} & 0 & G_{33} & G_{35} \\ F_{51}(\beta) & 0 & F_{53}(\beta) & F_{55}(\beta) \end{vmatrix}$$

$$+ \alpha_1 \begin{vmatrix} G_{11} & G_{12} & 0 & G_{15} \\ F_{21} & G_{12} & 0 & G_{15} \\ 0 & 0 & G_{34} & G_{35} \\ F_{51}(\beta) & 0 & 0 & F_{55}(\beta) \end{vmatrix} + \alpha_1\alpha_2 \begin{vmatrix} G_{11} & G_{13} & 0 & G_{15} \\ G_{31} & G_{33} & G_{34} & G_{35} \\ 0 & F_{43} & G_{34} & G_{35} \\ F_{51}(\beta) & F_{53}(\beta) & 0 & F_{55}(\beta) \end{vmatrix}$$

$$+ \alpha_2 \begin{vmatrix} G_{12} & 0 & 0 & G_{15} \\ 0 & G_{33} & G_{34} & G_{35} \\ 0 & F_{43} & G_{34} & G_{35} \\ 0 & F_{53}(\beta) & 0 & F_{55}(\beta) \end{vmatrix},$$

$$a_5 = a_5(\alpha_1, \alpha_2, \beta) = -\det A = -\alpha_1\alpha_2 \begin{vmatrix} G_{11} & G_{12} & G_{13} & 0 & G_{15} \\ F_{21} & G_{12} & 0 & 0 & G_{15} \\ G_{31} & 0 & G_{33} & G_{34} & G_{35} \\ 0 & 0 & F_{43} & G_{34} & G_{35} \\ F_{51}(\beta) & 0 & F_{53}(\beta) & 0 & F_{55}(\beta) \end{vmatrix}$$

$$= -\alpha_1\alpha_2 G_{12} G_{34} F_{55}(\beta)[(G_{11} - F_{21})(G_{33} - F_{43}) - G_{13}G_{31}].$$

On the basis of Assumption 2, we will arrange the coefficients of the characteristic equation (6) as polynomials with respect to the parameter β, expressing only their highest order terms explicitly. In these relations, we use the following notation:

$$d_1 = -\left(\frac{\partial r_1}{\partial M_1} + \frac{1}{E} \frac{\partial r_2}{\partial M_2} \right) > 0,$$

$$d_2 = -d_1(G_{12} + G_{34}) > 0,$$

$$d_3 = d_1 G_{11} + G_{15} \frac{\partial r_1}{\partial Y_1} > 0,$$

$$d_4 = d_1 G_{33} - G_{35} \frac{\partial r_2}{\partial Y_2} > 0,$$

$$d_5 = d_1 G_{12} G_{34} > 0\,,$$

$$d_6 = d_3 G_{34} - d_1 G_{12}(F_{21} - G_{11})\,,$$

$$d_7 = d_1 G_{34}(F_{43} - G_{33}) - d_4 G_{12},$$

$$d_8 = d_5(F_{21} - G_{11}) > 0,$$

$$d_9 = d_5(F_{43} - G_{33}) > 0,$$

$$d_{10} = d_1 G_{12} G_{34}[(F_{21} - G_{11})(F_{43} - G_{33}) - G_{13} G_{31}] > 0,$$

$$G = -\frac{\partial r_1}{\partial Y_1}[G_{15} G_{34}(F_{43} - G_{33}) + G_{12} G_{13} G_{35}] + \frac{\partial r_2}{\partial Y_2}[G_{12} G_{35}(F_{21} - G_{11}) + G_{15} G_{31} G_{34}]$$
$$- d_1[G_{12} G_{33}(F_{21} - G_{11}) + G_{11} G_{34}(F_{43} - G_{33}) + G_{13} G_{31}(G_{12} + G_{34})],$$

$$G^{(1)} = \frac{1}{d_2}(d_3 d_9 + d_4 d_8 + 2\sqrt{d_3 d_4 d_8 d_9}) > 0,$$

$$G^{(2)} = \frac{1}{d_2}(d_3 d_9 + d_4 d_8 - 2\sqrt{d_3 d_4 d_8 d_9}) > 0,$$

$$H = d_1(G_{11} G_{33} - G_{13} G_{31}) - \frac{\partial r_1}{\partial Y_1}(G_{13} G_{35} - G_{15} G_{33}) + \frac{\partial r_2}{\partial Y_2}(G_{15} G_{31} - G_{11} G_{35}).$$

The coefficients $a_j(\alpha_1, \alpha_2, \beta)$, $j = 1, 2, 3, 4, 5$, are given by the relations:

$$a_1(\alpha_1, \alpha_2, \beta) = d_1 \beta + f_1(\alpha_1, \alpha_2),$$

$$a_2(\alpha_1, \alpha_2, \beta) = (d_2 - d_3 \alpha_1 - d_4 \alpha_2)\beta + f_2(\alpha_1, \alpha_2),$$

$$a_3(\alpha_1, \alpha_2, \beta) = (d_5 + d_6 \alpha_1 - d_7 \alpha_2 + H \alpha_1 \alpha_2)\beta + f_3(\alpha_1, \alpha_2), \qquad (7)$$

$$a_4(\alpha_1, \alpha_2, \beta) = (d_8 \alpha_1 + d_9 \alpha_2 - G \alpha_1 \alpha_2)\beta + f_4(\alpha_1, \alpha_2),$$

$$a_5(\alpha_1, \alpha_2, \beta) = d_{10} \alpha_1 \alpha_2 \beta,$$

where $f_j(\alpha_1, \alpha_2)$, $j = 1, 2, 3, 4, 5$, are polynomials with respect to the parameters α_1, α_2 of order not higher than two.

The following theorem gives sufficient conditions for the existence of a critical triple of model (4).

Theorem 1. *If the inequalities*

$$G_{13}G_{31} < G_{11}G_{33},$$

$$\frac{d_3}{d_1(F_{21}-G_{11})} < \frac{G_{12}}{G_{34}} < \frac{d_1(F_{43}-G_{33})}{d_4}$$

are satisfied and

$$G > \frac{1}{d_2}(d_3 d_9 + d_4 d_8 + 2\sqrt{d_3 d_4 d_8 d_9}),$$

then there exist two critical triples $(\alpha_{10}, \alpha_{20}, \beta_0)$ *of model* (4).

Before the proof of Theorem 1, several lemmas will be introduced.

Lemma 1. *Characteristic equation (6) has two pairs of purely imaginary roots with the fifth one being negative if and only if the following relations are satisfied*

$$\begin{aligned} &\text{A.1} \quad a_1 > 0, a_2 > 0, a_4 > 0, \\ &\text{A.2} \quad a_2^2 - 4a_4 \geq 0, \\ &\text{A.3} \quad a_1 a_2 = a_3, \\ &\text{A.4} \quad a_1 a_4 = a_5. \end{aligned} \tag{8}$$

Proof. Denoting the roots of (6) as

$$\lambda_1 = i\omega_1, \lambda_2 = -i\omega_1, \lambda_3 = i\omega_2, \lambda_4 = -\mathrm{i}\omega_2, i = \sqrt{-1}, \lambda_5 < 0$$

and comparing (6) with its equivalent form

$$(\lambda - i\omega_1)(\lambda + i\omega_1)(\lambda - i\omega_2)(\lambda + i\omega_2)(\lambda - \lambda_5) = 0,$$

we get the assertion of lemma. □

Lemma 2. *The equations* A.3 *and* A.4 *from* (8) *can be expressed in the form*

$$\begin{aligned} &\text{A.3} \quad a_1 a_2 - a_3 = d_1(d_2 - d_3\alpha_1 - d_4\alpha_2)\beta^2 + g_1(\alpha_1, \alpha_2, \beta) = 0 \\ &\text{A.4} \quad a_1 a_4 - a_5 = d_1(d_8\alpha_1 + d_9\alpha_2 - G\alpha_1\alpha_2)\beta^2 + g_2(\alpha_1, \alpha_2, \beta) = 0, \end{aligned} \tag{9}$$

where for arbitrary fixed α_1 *and* α_2

$$\lim_{\beta \to \infty} \frac{g_j(\alpha_1, \alpha_2, \beta)}{\beta^2} = 0, \; j = 1,2.$$

Proof. If we set $a_j(\alpha_1,\alpha_2,\beta)$, j = 1, 2, 3, 4, 5, given by (7) into A.3 and A.4 in (8) and arrange the resulting equations as polynomials with respect to β, we obtain assertion (9). □

Consider the system of equations (10)–(11)

$$d_2 - d_3\alpha_1 - d_4\alpha_2 = 0, \tag{10}$$

$$d_8\alpha_1 + d_9\alpha_2 - G\alpha_1\alpha_2 = 0. \tag{11}$$

We are interested only in positive solutions α_1, α_2. Therefore, G must be positive. The first equation implies

$$\alpha_2 = \frac{d_2 - d_3\alpha_1}{d_4}, \tag{12}$$

which together with (11) means

$$d_8\alpha_1 + d_9\left(\frac{d_2 - d_3\alpha_1}{d_4}\right) - G\alpha_1\left(\frac{d_2 - d_3\alpha_1}{d_4}\right) = 0.$$

So, we obtain the following quadratic equation

$$d_3G\alpha_1^2 + (d_4d_8 - d_3d_9 - d_2G)\alpha_1 + d_2d_9 = 0. \tag{13}$$

The roots are determined by the formula

$$\alpha_1^{(1,2)} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \tag{14}$$

where

$$a = d_3G > 0, b = d_4d_8 - d_3d_9 - d_2G, c = d_2d_9 > 0. \tag{15}$$

The roots of (13) are positive if and only if $b < 0$ and $b^2 - 4ac \geq 0$. From (15) we see that $b < 0$ when

$$G > \frac{d_4d_8 - d_3d_9}{d_2}. \tag{16}$$

The inequality $b^2 - 4ac \geq 0$ is satisfied if and only if

$$d_2^2G^2 - 2d_2(d_3d_9 + d_4d_8)G + (d_3d_9 - d_4d_8)^2 \geq 0.$$

Since $G > 0$, this inequality is satisfied for

$$G \in (0, G^{(2)}\rangle \cup \langle G^{(1)}, \infty),$$

where

$$G^{(1,2)} = \frac{d_3 d_9 + d_4 d_8 \pm 2\sqrt{d_3 d_4 d_8 d_9}}{d_2}.$$

Lemma 3. *The system of equations* (10)–(11)

a) *has two positive solutions* $(\alpha_1^{(1,2)}, \alpha_2^{(1,2)})$ *for* $G \in (G^{(1)}, \infty)$.

b) *has one positive solution* (α_1^*, α_2^*) *for* $G = G^{(1)}$,

c) *has no positive solutions for* $G \in (0, G^{(2)}\rangle$.

Proof. Let $G \in (0, G^{(2)}\rangle \cup \langle G^{(1)}, \infty)$. Then (13) has real roots $\alpha_1^{(1)}, \alpha_1^{(2)}$ with the same sign. From (12) we obtain real values $\alpha_2^{(1,2)}$

$$\alpha_2^{(2)} = \frac{d_2 - d_3 \alpha_1^{(2)}}{d_4} \geq \alpha_2^{(1)} = \frac{d_2 - d_3 \alpha_1^{(1)}}{d_4}. \tag{17}$$

We have

$$\alpha_2^{(1)} \alpha_2^{(2)} = \frac{d_2 - d_3 \alpha_1^{(1)}}{d_4} \cdot \frac{d_2 - d_3 \alpha_1^{(2)}}{d_4} = \frac{d_2^2 - d_2 d_3 (\alpha_1^{(1)} + \alpha_1^{(2)}) + d_3^2 \alpha_1^{(1)} \alpha_1^{(2)}}{d_4^2}. \tag{18}$$

Since

$$\alpha_1^{(1)} \alpha_1^{(2)} = \frac{c}{a} = \frac{d_2 d_9}{d_3 G}$$

and

$$\alpha_1^{(1)} + \alpha_1^{(2)} = \frac{-b}{a} = \frac{d_3 d_9 - d_4 d_8 + d_2 G}{d_3 G},$$

we have

$$\alpha_2^{(1)} \alpha_2^{(2)} = \frac{d_2^2 - d_2 d_3 \dfrac{d_3 d_9 - d_4 d_8 + d_2 G}{d_3 G} + d_3^2 \dfrac{d_2 d_9}{d_3 G}}{d_4^2} = \frac{d_2 d_8}{d_4 G} > 0. \tag{19}$$

Which means that both values have the same sign. So, consider

$$\alpha_2^{(1)} + \alpha_2^{(2)} = \frac{d_2 - d_3 \alpha_1^{(1)}}{d_4} + \frac{d_2 - d_3 \alpha_1^{(2)}}{d_4} = \frac{2d_2 - d_3 (\alpha_1^{(1)} + \alpha_1^{(2)})}{d_4}$$

$$= \frac{2d_2 - d_3 \dfrac{d_3 d_9 - d_4 d_8 + d_2 G}{d_3 G}}{d_4} = \frac{d_4 d_8 - d_3 d_9 + d_2 G}{d_4 G}.$$

Both values $\alpha_2^{(1)}$, $\alpha_2^{(2)}$ are positive if and only if

$$d_4 d_8 - d_3 d_9 + d_2 G > 0 . \qquad (20)$$

Conditions (16) and (20) are satisfied if and only if

$$d_2 G > | d_4 d_8 - d_3 d_9 | . \qquad (21)$$

If

$$G \ge G^{(1)} = \frac{d_3 d_9 + d_4 d_8 + 2\sqrt{d_3 d_4 d_8 d_9}}{d_2} > \frac{d_4 d_8 + d_3 d_9}{d_2},$$

then (21) is satisfied and all four values $\alpha_{1,2}^{(1,2)}$ are positive. If $0 < G \le G^{(2)}$, then

$$G \le \frac{d_3 d_9 + d_4 d_8 - 2\sqrt{d_3 d_4 d_8 d_9}}{d_2} = \frac{|\sqrt{d_3 d_9} - \sqrt{d_4 d_8}|^2}{d_2}.$$

Since

$$\frac{|\sqrt{d_3 d_9} - \sqrt{d_4 d_8}|^2}{d_2} < \frac{|\sqrt{d_3 d_9} - \sqrt{d_4 d_8}||\sqrt{d_3 d_9} + \sqrt{d_4 d_8}|}{d_2} = \frac{|d_3 d_9 - d_4 d_8|}{d_2},$$

(21) is not satisfied and thus either $\alpha_1^{(1)} < 0$, $\alpha_1^{(2)} < 0$ or $\alpha_2^{(1)} < 0$, $\alpha_2^{(2)} < 0$. □

Lemma 4. *For any* $G \in (G^{(1)}, \infty)$ *there exists* $\beta_G > 0$ *such that for any* $\beta > \beta_G$ *there are two positive solutions* $(\alpha_{10}^{(i)}, \alpha_{20}^{(i)})$, $i = 1,2$, of system (9).

Proof. Instead of β we will consider $\gamma = \dfrac{1}{\beta}$. Set

$$\Phi(\alpha_1, \alpha_2, \gamma) = \begin{cases} (d_2 - d_3\alpha_1 - d_4\alpha_2) + \dfrac{\gamma^2}{d_1} g_1(\alpha_1, \alpha_2, \dfrac{1}{\gamma}) & \text{for } \gamma \ne 0 \\ (d_2 - d_3\alpha_1 - d_4\alpha_2) & \text{for } \gamma = 0 . \end{cases}$$

$$\Psi(\alpha_1, \alpha_2, \gamma) = \begin{cases} (d_8\alpha_1 + d_9\alpha_2 - G\alpha_1\alpha_2) + \dfrac{\gamma^2}{d_1} g_2(\alpha_1, \alpha_2, \dfrac{1}{\gamma}) & \text{for } \gamma \ne 0 \\ (d_8\alpha_1 + d_9\alpha_2 - G\alpha_1\alpha_2) & \text{for } \gamma = 0 . \end{cases}$$

The functions Φ and Ψ are polynomials in α_1, α_2 and γ. System (9) is equivalent to the system

$$\Phi(\alpha_1,\alpha_2,\gamma)=0$$

$$\Psi(\alpha_1,\alpha_2,\gamma)=0\,. \tag{22}$$

For $\gamma=0$ we obtain the system (10)–(11), which has two positive solutions $(\alpha_1^{(i)},\alpha_2^{(i)})$ for any $G\in(G^{(1)},\infty)$. Let α_1 and α_2 satisfy (10)–(11). Consider the Jacobian

$$\begin{vmatrix} \dfrac{\partial\Phi}{\partial\alpha_1}(\alpha_1,\alpha_2,0) & \dfrac{\partial\Phi}{\partial\alpha_2}(\alpha_1,\alpha_2,0) \\ \dfrac{\partial\Psi}{\partial\alpha_1}(\alpha_1,\alpha_2,0) & \dfrac{\partial\Psi}{\partial\alpha_2}(\alpha_1,\alpha_2,0) \end{vmatrix} = \begin{vmatrix} -d_3\,, & -d_4 \\ d_8-G\alpha_2\,, & d_9-G\alpha_1 \end{vmatrix}$$

$$=d_4d_8-d_3d_9+G(d_3\alpha_1-d_4\alpha_2)=d_4d_8-d_3d_9+G(2d_3\alpha_1-d_2).$$

The Jacobian is zero only for

$$\alpha_1=\frac{d_3d_9-d_4d_8+Gd_2}{2d_3G}=\frac{-b}{2a}\,.$$

The Jacobian is nonzero at both points $(\alpha_1^{(i)},\alpha_2^{(i)})$. System (22) has two solutions for γ sufficiently close to 0. □

Remark 3. For $G=G^{(1)}$ we have one positive solution, because $\alpha_1^{(1)}=\alpha_1^{(2)}$ and $\alpha_2^{(1)}=\alpha_2^{(2)}$. However, the Jacobian is zero and hence the Implicit Function Theorem cannot be used.

Lemma 5. *If the inequalities*

$$\frac{d_3}{d_1(F_{21}-G_{11})}<\frac{G_{12}}{G_{34}}<\frac{d_1(F_{43}-G_{33})}{d_4}\,, \tag{23}$$

$$G_{13}G_{31}<G_{11}G_{33} \tag{24}$$

are satisfied, then $a_3(\alpha_1,\alpha_2,\beta)$ *from* (7) *is positive for an arbitrary pair* (α_1,α_2) *of parameters* α_1, α_2 *and sufficiently large parameter* β.

Proof. The coefficient $a_3(\alpha_1,\alpha_2,\beta)$ is given by the relation

$$a_3(\alpha_1,\alpha_2,\beta)=(d_5+d_6\alpha_1-d_7\alpha_2+H\alpha_1\alpha_2)\beta+f_3(\alpha_1,\alpha_2),$$

where:

$$d_6 = d_3 G_{34} - d_1 G_{12}(F_{21} - G_{11}),$$

$$d_7 = d_1 G_{34}(F_{43} - G_{33}) - d_4 G_{12},$$

$$H = d_1(G_{11}G_{33} - G_{13}G_{31}) - \frac{\partial r_1}{\partial Y_1}(G_{13}G_{35} - G_{15}G_{33}) + \frac{\partial r_2}{\partial Y_2}(G_{15}G_{31} - G_{11}G_{35}).$$

According to (23) d_6 is positive, d_7 negative and according to (24) H is positive. Therefore, $a_3(\alpha_1, \alpha_2, \beta)$ is positive for an arbitrary pair of values (α_1, α_2) of parameters α_1, α_2 and sufficiently large value of parameter β. □

Proof of Theorem 1. Lemma 4 guarantees the existence of two triples $(\alpha_1^{(i)}, \alpha_2^{(i)}, \beta)$ of parameters α_1, α_2 and β, which satisfy conditions A.3 and A.4 from Lemma 1. The condition $a_1(\alpha_1^{(i)}, \alpha_2^{(i)}, \beta) > 0$ is satisfied for sufficiently large β. Lemma 5 guarantees the positiveness of $a_3(\alpha_1^{(i)}, \alpha_2^{(i)}, \beta)$ for sufficiently large β. As $a_1(\alpha_1^{(i)}, \alpha_2^{(i)}, \beta) > 0$, $a_3(\alpha_1^{(i)}, \alpha_2^{(i)}, \beta) > 0$ and $a_5(\alpha_1^{(i)}, \alpha_2^{(i)}, \beta) > 0$ then from A.3 and A.4 the positiveness of $a_2(\alpha_1^{(i)}, \alpha_2^{(i)}, \beta)$ and $a_4(\alpha_1^{(i)}, \alpha_2^{(i)}, \beta)$ follows for sufficiently large β. In this way, the condition A.1 in Lemma 1 is satisfied. It is clear that condition A.2 is also satisfied for sufficiently large β. Therefore, there exist two critical triples $(\alpha_{10}^{(i)}, \alpha_{20}^{(i)}, \beta_0)$, $i = 1,2$, of model (3). □

Now take an arbitrary critical triple $(\alpha_{10}, \alpha_{20}, \beta_0)$ of model (4), fix β_0 and investigate model (4) with respect to parameters α_1 and α_2, which are connected by $\alpha_2 = \kappa(\alpha_1 - \alpha_{10}) + \alpha_{20}$, where κ is a real constant and $\alpha_1 \in (\alpha_{10} - \delta, \alpha_{10} + \delta)$, where $\delta > 0$. After shifting the critical values α_{10}, α_{20} to the origin by the translation $\widetilde{\alpha}_1 = \alpha_1 - \alpha_{10}$, $\widetilde{\alpha}_2 = \alpha_2 - \alpha_{20}$, denoting $\varepsilon = \widetilde{\alpha}_1$ and performing a linear mapping $y = Tx$, which transforms the matrix $A(\alpha_{10}, \alpha_{20}, \beta_0)$ into its Jordan form, model (4) takes the form

$$\dot{x} = Jx + X(x, \varepsilon, \kappa), \tag{25}$$

where $x = (x_1, x_2, x_3, x_4, x_5)$, $x_2 = \overline{x}_1$, $x_4 = \overline{x}_3$,

$$J = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 & 0 \\ 0 & 0 & 0 & \lambda_4 & 0 \\ 0 & 0 & 0 & 0 & \lambda_5 \end{pmatrix}, \quad X = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \\ X_5 \end{pmatrix}, \quad X_2 = \overline{X}_1, \ X_4 = \overline{X}_3,$$

and the sign "$\overline{}$" denotes the complex conjugate.

On the basis of Lemma 1 in [4, pp. 22–23], we can formulate the following lemma.

Lemma 6. *If* $p_1\omega_1 + p_2\omega_2 \neq 0$ *for* $0 < |p_1| + |p_2| \leq 5$, $p_1, p_2 \in \mathbb{Z}$, *then there exists a transformation*

$$x_j = u_j + h_j(u_1, u_2, u_3, u_4, \varepsilon, \kappa),\ j = 1, 2, 3, 4, 5,$$

$$h_j = \sum_{\substack{m_1+m_2+m_3+m_4+m \geq 2 \\ m\in\{0,1\}}}^{4-2m} c_j^{(m_1,m_2,m_3,m_4,m)} u_1^{m_1} u_2^{m_2} u_3^{m_3} u_4^{m_4} \varepsilon^m, \tag{26}$$

which transforms model (25) *into the form*

$$\begin{aligned}
\dot{u}_1 &= \lambda_1 u_1 + \zeta_1(\kappa)u_1\varepsilon + \eta_1 u_1^2 u_2 + \chi_1 u_1 u_3 u_4 \\
&\quad + U_1^0(u_1,u_2,u_3,u_4,u_5,\varepsilon,\kappa) + U_1^*(u_1,u_2,u_3,u_4,u_5,\varepsilon,\kappa), \\
\dot{u}_2 &= \lambda_2 u_2 + \zeta_2(\kappa)u_2\varepsilon + \eta_2 u_1 u_2^2 + \chi_2 u_2 u_3 u_4 \\
&\quad + U_2^0(u_1,u_2,u_3,u_4,u_5,\varepsilon,\kappa) + U_2^*(u_1,u_2,u_3,u_4,u_5,\varepsilon,\kappa), \\
\dot{u}_3 &= \lambda_3 u_3 + \zeta_3(\kappa)u_3\varepsilon + \eta_3 u_3^2 u_4 + \chi_3 u_3 u_1 u_2 \\
&\quad + U_3^0(u_1,u_2,u_3,u_4,u_5,\varepsilon,\kappa) + U_3^*(u_1,u_2,u_3,u_4,u_5,\varepsilon,\kappa), \\
\dot{u}_4 &= \lambda_4 u_4 + \zeta_4(\kappa)u_4\varepsilon + \eta_4 u_3 u_4^2 + \chi_4 u_4 u_1 u_2 \\
&\quad + U_4^0(u_1,u_2,u_3,u_4,u_5,\varepsilon,\kappa) + U_4^*(u_1,u_2,u_3,u_4,u_5,\varepsilon,\kappa), \\
\dot{u}_5 &= \lambda_5 u_5 + U_5^0(u_1,u_2,u_3,u_4,u_5,\varepsilon,\kappa) + U_5^*(u_1,u_2,u_3,u_4,u_5,\varepsilon,\kappa),
\end{aligned} \tag{27}$$

where λ_j *are the eigenvalues of J,* $u_2 = \overline{u_1}$, $u_4 = \overline{u_3}$, $U_j^0(u_1,u_2,u_3,u_4,0,\varepsilon,\kappa) = 0$, $U_j^*(\sqrt{\varepsilon}u_1,\sqrt{\varepsilon}u_2,\sqrt{\varepsilon}u_3,\sqrt{\varepsilon}u_4,\sqrt{\varepsilon}u_5,\varepsilon,\kappa) = (\sqrt{\varepsilon})^5 \tilde{U}_j(u_1,u_2,u_3,u_4,u_5,\varepsilon,\kappa), j = 1, 2, 3, 4, 5.$

In polar coordinates

$$u_1 = \rho_1 e^{i\vartheta_1}, u_2 = \rho_1 e^{-i\vartheta_1}, u_3 = \rho_2 e^{i\vartheta_2}, u_4 = \rho_2 e^{-i\vartheta_2}, u_5 = v,$$

model (27) is of the form

$$\dot{\rho}_1 = \rho_1(a_{11}\rho_1^2 + a_{12}\rho_2^2 + q_1\varepsilon) + R_1^0(\rho_1,\vartheta_1,\rho_2,\vartheta_2,v,\varepsilon) + R_1^*(\rho_1,\vartheta_1,\rho_2,\vartheta_2,v,\varepsilon),$$

$$\dot{\rho}_2 = \rho_2(a_{21}\rho_1^2 + a_{22}\rho_2^2 + q_2\varepsilon) + R_2^0(\rho_1,\vartheta_1,\rho_2,\vartheta_2,v,\varepsilon) + R_2^*(\rho_1,\vartheta_1,\rho_2,\vartheta_2,v,\varepsilon),$$

$$\dot{\vartheta}_1 = \omega_1 + b_{11}\rho_1^2 + b_{12}\rho_2^2 + s_1\varepsilon + \frac{1}{\rho_1}[\Theta_1^0(\rho_1,\vartheta_1,\rho_2,\vartheta_2,v,\varepsilon) + \Theta_1^*(\rho_1,\vartheta_1,\rho_2,\vartheta_2,v,\varepsilon)], \quad (28)$$

$$\dot{\vartheta}_2 = \omega_2 + b_{21}\rho_1^2 + b_{22}\rho_2^2 + s_2\varepsilon + \frac{1}{\rho_2}[\Theta_2^0(\rho_1,\vartheta_1,\rho_2,\vartheta_2,v,\varepsilon) + \Theta_2^*(\rho_1,\vartheta_1,\rho_2,\vartheta_2,v,\varepsilon)],$$

$$\dot{v} = \lambda_5 v + V^0(\rho_1,\vartheta_1,\rho_2,\vartheta_2,v,\varepsilon) + V^*(\rho_1,\vartheta_1,\rho_2,\vartheta_2,v,\varepsilon),$$

where $a_{11} = \mathrm{Re}\,\eta_1$, $a_{12} = \mathrm{Re}\,\chi_1$, $a_{21} = \mathrm{Re}\,\chi_3$, $a_{22} = \mathrm{Re}\,\eta_3$, $q_1 = \mathrm{Re}\,\zeta_1$, $q_2 = \mathrm{Re}\,\zeta_3$, $b_{11} = \mathrm{Im}\,\eta_1$, $b_{12} = \mathrm{Im}\,\chi_1$, $b_{21} = \mathrm{Im}\,\chi_3$, $b_{22} = \mathrm{Im}\,\eta_3$, $s_1 = \mathrm{Im}\,\zeta_1$, $s_2 = \mathrm{Im}\,\zeta_3$ and the functions with superscripts "0" and "*" have analogous properties to the functions with such superscripts in model (27).

The equation

$$\boldsymbol{B}\rho^2 + \varepsilon\, q = 0\,, \quad (29)$$

where

$$\boldsymbol{B} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \rho^2 = \begin{pmatrix} \rho_1^2 \\ \rho_2^2 \end{pmatrix}, q = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$$

is the bifurcation equation of model (28). Suppose that $\det \boldsymbol{B} \neq 0$. Denote the solution of (29) as

$$\rho^2 = \varepsilon \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

and consider the matrix

$$\boldsymbol{P} = \begin{pmatrix} a_{11}\,|\,w_1\,| & a_{12}\sqrt{w_1 w_2} \\ a_{21}\sqrt{w_1 w_2} & a_{22}\,|\,w_2\,| \end{pmatrix}.$$

From Theorem 1 in [4, p. 25], we can formulate the following Theorem.

Theorem 2. *Suppose that the coordinates* w_1 *and* w_2 *have the same sign. If the eigenvalues of matrix* $\boldsymbol{P}$ *are not purely imaginary, then model* (28) *has an invariant torus for every* $\varepsilon \in (0,\varepsilon_0)$, *– if* $w_1 > 0$, *and for every* $\varepsilon \in (-\varepsilon_0, 0)$, *if* $w_1 < 0$, *where* ε_0 *is sufficiently small. The torus is given by the equations*

$$\rho_1 = \sqrt{\varepsilon w_1} + \sqrt{|\varepsilon^3|}\, f_1(\vartheta_1, \vartheta_2, \varepsilon),$$

$$\rho_2 = \sqrt{\varepsilon w_2} + \sqrt{|\varepsilon^3|}\, f_2(\vartheta_1, \vartheta_2, \varepsilon),$$

$$v = \sqrt{|\varepsilon^5|}\, g(\vartheta_1, \vartheta_2, \varepsilon)$$

where f_1, f_2 and g are continuous functions with respect to ϑ_1, ϑ_2, and ε for arbitrary ϑ_1, ϑ_2 and $\varepsilon \in (0, \varepsilon_0)$, resp. $\varepsilon \in (-\varepsilon_0, 0)$, and 2π-periodic in ϑ_1, ϑ_2.

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Czy torusy mogą się pojawiać w modelu dwuregionalnym przy ustalonej stopie wymiany?

W artykule omówiono dwuregionalny model wprowadzony przez T. Asada [1], opisujący dynamiczne wzajemne oddziaływanie dwóch regionów połączonych poprzez międzyregionalny handel i przepływ kapitału. Model pokazuje rozwój dochodu, akcji kapitałowych i akcji pieniężnych w rozważanych regionach. T. Asada [1] dokonał analizy istnienia punktu równowagi dla modelu, znalazł warunki wystarczające dla jego lokalnej stabilności, a także przeanalizował problem istnienia cykli biznesowych wokół punktu równowagi. Ponieważ badanym modelem jest pięciowymiarowy dynamiczny system, problem istnienia torusa wokół punktu równowagi jest uzasadniona. Artykuł daje odpowiedź na to pytanie. Torusy mogą pojawiać się tylko w przypadku, kiedy macierz aproksymacji liniowej modelu w punkcie równowagi ma dwie pary czysto urojonych wartości własnych. Twierdzenie 1 daje wystarczające warunki istnienia dwóch par czysto urojonych wartości własnych z pozostałą jedną wartością ujemną. Twierdzenie 2 stanowi komentarz istnienia torusów w bliskim sąsiedztwie punktu równowagi. Model rozważany w artykule może być zastosowany do analizy dynamicznego wzajemnego oddziaływania dwóch krajów w obszarze strefy Euro.

Słowa kluczowe: *model dynamiczny, równowaga, macierz przybliżenia liniowego, wartości własne macierzy, normalna forma równań różniczkowych na niezmienniczej powierzchni, równania bifurkacyjne, torus*