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DEPENDENT DISCRETE RISK PROCESSES – CALCULATION OF THE PROBABILITY OF RUIN[1]

This paper is devoted to discrete processes of dependent risks. The random variables describing the time between claims can be dependent in such processes, unlike under the classical approach. The ruin problem is investigated and the probably of ruin is computed. The relation between the degree of dependence and the probability of ruin is studied.

Three cases are presented. Different methods of characterizing the dependency structure are examined. First, strictly dependent times between claims are investigated. Next, the dependency structure is described using an Archimedean copula or using Markov chains. In the last case, three situations in which the probability of ruin can be exactly computed are presented. Numerical examples in which the claims have a geometric distribution are investigated. A regular relation between the probability of ruin and the degree of dependence is only observed in the Markov chain case.

Keywords: *risk process, probability of ruin, dependence, copula, Markov chain*

1. Introduction

This paper is devoted to dependent discrete risk processes. In this paper, we weaken the strong classical assumption of independence of the times between the occurrences of claims. We admit dependence of the binary random variables I_j, which determine the moments at which claims occur. The assumption of independence is very useful from a scientific point of view, but it is often unrealistic. In practice, many variables and processes are dependent.

The dependent risk processes described in our paper are studied from the point of view of ruin theory. We are interested in determining the probability of ruin and study

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the influence of the degree of dependence on the value of this probability. We consider different kinds of dependency structure for the random variables I_j.

First, we recall the classical risk model introduced by SHIU in [9], in which the random variables I_j are independent. We present cases where we can compute the probability of ruin exactly. Next, we introduce the extreme case of strict dependence between the occurrence of claims and the case where the dependency structure is described by an Archimedean copula. An Archimedean copula induces random variables, which can be interpreted as external factors. These factors influence all risks to the same degree. The last section of the paper is devoted to a model in which the random variables I_j create a stationary Markov chain with a binary state space.

2. Presentation of the problem

We will investigate the following discrete risk process in our paper:

$$U(t) = u + t - \sum_{i=1}^{t} Y_i ,$$

where $t = 1, 2, ...$, u is the initial capital, the values of the claims $Y_i = I_i X_i$, $X_i = 1, 2, ...$ are discrete random variables and

$$I_i = \begin{cases} 1 & \text{with probability } q, \\ 0 & \text{with probability } p, \end{cases}$$

where $p = 1 - q$, is a binary random variable indicating the occurrence of a claim and independent of the variables X_i. We assume that $U(0) = u$.

Let us assume that the values of claims are independent and identically distributed:

$$P(X_i = k) = f(k),$$

$k = 1, 2, ...$, with cumulative distribution function

$$F(n) = \sum_{k=1}^{n} f(k)$$

and expected value

$$m = E(X_i) = \sum_{k=1}^{\infty} k f(k) .$$

We assume that the binary random variables I_i, the indicators reflecting the existence of a claim, need not be independent, in contrast to classical risk models [5], [9]. This assumption also implies that the random variables Y_i may be dependent.

We will describe the event that the risk process takes a negative value at some time as ruin within an infinite time horizon, in short ruin. The probability of ruin depends on the initial capital u and is determined by the formula:

$$\psi(u) = P(U(t) < 0 \text{ for some } t \mid U(0) = u).$$

The survival function is the probability that ruin does not occur and takes the form:

$$\phi(u) = 1 - \psi(u).$$

GERBER in [5] defines the probability of ruin as

$$\psi^*(u) = P(U(t) \le 0 \text{ for some } t \mid U(0) = u).$$

This approach is not essentially different from ours, based on the definition by SHIU [9], because we obtain the following relation between these probabilities [3]:

$$\psi^*(u) = \psi(u-1),$$

for $u = 1, 2, ...$

3. Independence

Let us assume that the random variables I_i are independent. This is a classical approach to a ruin problem [3,9]. The moments of the appearances of claims are independent in this case. If $qm \ge 1$, then ruin is a certain event for any value of initial capital u, i.e. $\psi(u) = 1$. So we assume that the opposite relation holds:

$$qm < 1,$$

in other words, there exists a relative security loading $\eta > 0$ such that

$$(1 + \eta)qm = 1.$$

In this case, we can derive the probability of ruin $\psi_I(u)$ using the following recursive formulas [9], [2]:

$$\psi_I(0) = \frac{q}{p}(m-1),$$

$$\psi_I(u) = \psi_I(0) - \frac{q}{p}\sum_{k=1}^{u}(1-F(k))(1-\psi_I(u-k)),$$

where $k = 1, 2, ...$, or the following formulas based on the survival functions

$$\phi_I(0) = \frac{1-qm}{p},$$

$$\phi_I(u) = \frac{1}{p}\left(\phi_I(u-1) - q\sum_{k=1}^{u}\phi_I(u-k)f(k)\right).$$

In the limit when $u \to \infty$, the probability of ruin tends to zero:

$$\psi_I(\infty) = 0 .$$

In some cases, we can calculate the exact value of the probability of ruin. Now, we introduce three types of random variable X_i representing the value of a claim [1], for which there exists an algebraic formula for the probability of ruin.

a) deterministic variable: $P(X_i = x) = 1$

If $x = 1$, then $\psi_I(u) = 0$ for $u \geq 0$. On the other hand, when $x = 2$, we obtain

$$\psi_I(u) = \left(\frac{q}{p}\right)^{u+1},$$

for $u \geq 0$ and $q < 0.5$, and $\psi_I(u) = 1$ for $q \geq 0.5$.

b) two-point distribution

Let X_i be a random variable with support $\{1, 2\}$. Then the expected value of a claim is equal to $m = 1 + f(2)$, and the probability of ruin takes the form

$$\psi_I(u) = \left(\frac{qf(2)}{p}\right)^{u+1}$$

for $u \geq 0$ and $q < \frac{1}{1+f(2)}$. If $f(2) = 1$, then we obtain case a).

c) geometric distribution

We obtain an exact formula for the probability of ruin when the claims X_i have a geometric distribution:

$$f(k) = (1-\beta)\beta^{k-1},$$

where $k = 1, 2, \ldots$ with expected value

$$m = \frac{1}{1-\beta}.$$

Then the probability of ruin is equal to

$$\psi_I(u) = \frac{q}{1-\beta}\left(\frac{\beta}{p}\right)^{u+1},$$

for $u \geq 0$ and $q < 1-\beta$.

4. Strict dependence

Let us assume that the binary random variables I_i are strictly dependent. Either all of them are equal to 1 or all are equal to 0. The sum of the random variables Y_i is equal to

$$\sum_{i=1}^{t} Y_i = \begin{cases} X_1 + \ldots + X_t & \text{with probability } q, \\ 0 & \text{with probability } p \end{cases}$$

in this case. When $I_j = 0$, there are no claims and ruin does not occur. Otherwise, if $m = 1$, then the claims are always equal to 1 and there is no ruin either, because $U(t) = u \geq 0$ for all t. However, for $m > 1$ and sufficiently large values of t, the expected value $EU(t) = u + t(1 - m) < 0$ and ruin occurs with probability 1. So the probability of ruin, $\psi_c(u)$, does not depend on the value of initial capital $u \geq 0$, and is equal to

$$\psi_c(u) = \begin{cases} q & \text{for } m > 1, \\ 0 & \text{for } m = 1. \end{cases}$$

We see that when the claims are not all equal to 1, then the probability of ruin is greater than zero independently of the value of initial capital u, i.e. we obtain the following relation

$$\psi_I(\infty) < \psi_c(\infty),$$

for $m > 1$.

When the initial capital $u = 0$, we observe a more complicated situation. The inequality

$$\frac{q}{p}(m-1) > q$$

implies that for $m > 1$ we have

$$\psi_c(0) < \psi_I(0) \qquad \text{for } m + q > 2,$$

$$\psi_c(0) = \psi_I(0) \qquad \text{for } m + q = 2,$$

$$\psi_c(0) > \psi_I(0) \qquad \text{for } m + q < 2.$$

When $m + q < 2$, the probability of ruin in the case of strict dependence, $\psi_c(u)$, is greater than in the case of independence for all values of initial capital, i.e.

$$\psi_c(u) > \psi_I(u)$$

for all $u > 0$. This simply results from the fact that the function $\psi_c(u)$ is a constant function and $\psi_I(u)$ is a decreasing function. When $m + q > 2$, for relatively small values of initial capital the probability of ruin in the case of independence is greater than the probability of ruin in the case of strict dependence. Also, we obtain the inequality $\psi_I(u) < \psi_c(u)$ for values of initial capital greater than some u_0.

Example 1. Let us assume that claims occur with probability $q = 0.2$ and they have a geometric distribution with parameter $\beta = 2/3$. Thus the expected value of a claim is 3. In the case of independent claims the probability of ruin is equal to

$$\psi_I = 0.6\left(\frac{5}{6}\right)^{u+1},$$

and in the case of strictly dependent claims, this probability is equal to 0.2 for all values of initial capital u. The probability of ruin in the cases of independent and strictly dependent claims is presented in fig. 1 and for strict dependent claims this probability is equal to 0,2 for all values of initial capital u. The probability of ruin for the independent and for strict dependent claims are presented in fig. 1.

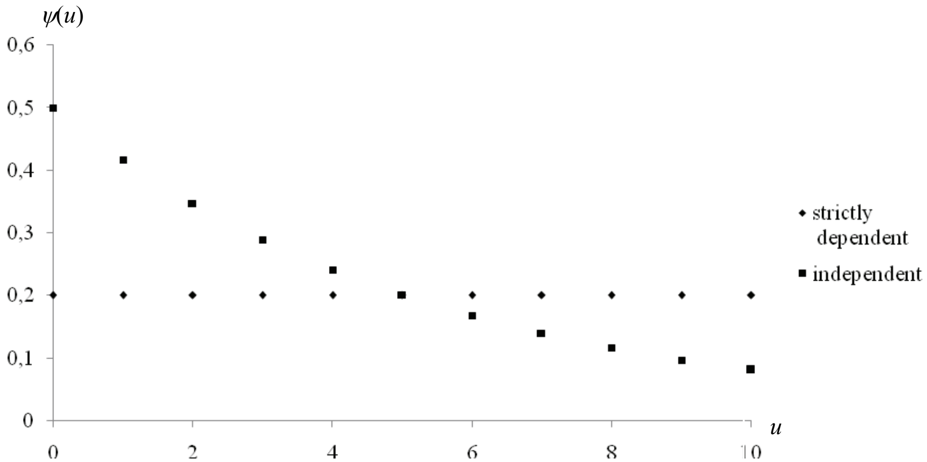

Fig. 1. The probability of ruin for independent and strictly dependent claims
Source: Author's own work.

5. Archimedean copulas

Now we assume that the dependence structure between the binary random variables I_i is described by a copula C. The copula C is the link between the joint and marginal distributions. In some cases, we can determine the copula using survival functions: joint $\overline{H}(x_1, ..., x_n) = P(I_1 > x_1, ..., I_n > x_n)$ and marginal

$$\overline{H}_i(x) = P(I_i > x) = \begin{cases} 0 & \text{for } x \geq 1, \\ q & \text{for } 0 \leq x < 1, \\ 1 & \text{for } x < 1. \end{cases}$$

We use such an approach in our paper. Hence, we define the copula C using the formula

$$\overline{H}(x_1, ..., x_n) = C(\overline{H}_1(x_1), ..., \overline{H}_n(x_n)) .$$

Archimedean copulas have a simple, quasi-additive form. They are generated by a one-dimensional function g, called a generator, using the formula

$$C(u_1, ..., u_n) = g^{-1}(g(u_1) + ... + g(u_n)),$$

where $u_i \in [0, 1]$, g: $(0, 1] \to R_+$ is a continuous, decreasing function, such that $\lim_{u \to 0} g(u) = \infty$ and $g(1) = 0$. The function C is well defined by the above formula for all $n \geq 2$ iff g^{-1} is a complete monotonic function on $[0, \infty)$ (see theorem 4.6.2 in [7]), i.e. the function g satisfies the following condition for all $s \geq 0$ and $k = 1, 2, ...$

$$(-1)^k \frac{d^k}{dx^k} g^{-1}(s) \geq 0 .$$

The function $g^{-1}(s)$ is complete monotonic, so it is the Laplace transform of some non-negative random variable Θ [7]:

$$g^{-1}(s) = L_\Theta(s).$$

Thus the indicators I_i are conditionally independent for a fixed value of the random variable Θ [5, 6, 8], i.e.

$$\overline{H}(x_1, ..., x_n \mid \theta) = P(I_1 > x_1, ..., I_n > x_n \mid \Theta = \theta)$$
$$= P(I_1 > x_1 \mid \Theta = \theta) ... P(I_n > x_n \mid \Theta = \theta) = \overline{H}_1(x_1 \mid \theta) ... \overline{H}_n(x_n \mid \theta).$$

Moreover, the conditional, marginal survival functions are determined by the generator g of some Archimedean copula using formula [8]

$$\overline{H}_i(x \mid \theta) = \exp(-\theta g(\overline{H}_i(x))) .$$

However, we can present the joint and marginal survival functions of the random variables I_i as the following mixture:

$$\overline{H}(x_1, ..., x_n) = \int_0^\infty \overline{H}(x_1, ..., x_n \mid \theta) dF_\Theta(\theta),$$

$$\overline{H}_i(x) = \int_0^\infty \overline{H}_i(x \mid \theta) dF_\Theta(\theta),$$

where $F_\Theta(\theta)$ is the cumulative distribution function of the random variable Θ.

In this case, the conditional expected value is equal to

$$\overline{H}_i(x \mid \theta) = \begin{cases} 0 & \text{for } x \geq 1, \\ \exp(-\theta g(q)) & \text{for } 0 \leq x < 1, \\ 1 & \text{for } x < 1. \end{cases}$$

We can treat the expression $\exp(-\theta g(q))$ as the conditional probability of ruin and we denote it by $q(\theta)$.

Conditioning on a fixed value θ of the random variable Θ, we obtain the binary random variables $I_{i|\theta}$. These random variables induce the conditional risk process $U_\theta(u)$. Hence, we can investigate the conditional probability of ruin $\psi(u|\theta)$ for any value of initial capital u. The function $\psi(u|\theta)$ is decreasing with respect to θ, because $q(\theta)$ is also a decreasing function. In this case, the unconditional probability of ruin is a mixture of the conditional probabilities:

$$\psi(u) = \int_0^\infty \psi(u \mid \theta) dF_\Theta(\theta).$$

Let

$$\theta_0 = \frac{\ln m}{g(q)},$$

then for every $\theta \leq \theta_0$ the appearance of conditional ruin is a certain event, i.e. $\psi(u|\theta) = 1$ and we obtain the following equality:

$$q(\theta_0) = \frac{1}{m}.$$

We can derive the unconditional probability of ruin using the following formula:

$$\psi(u) = \int_{\theta_0}^\infty \psi(u \mid \theta) dF_\Theta(\theta) + F_\Theta(\theta_0).$$

If the initial capital is equal to zero, then the unconditional probability of ruin takes the form:

$$\psi(0) = \int_{\theta_0}^{\infty} \frac{q(\theta)}{1-q(\theta)}(m-1)dF_\Theta(\theta) + F_\Theta(\theta_0) \,,$$

and for an infinite value of initial capital we obtain $\psi(\infty) = F_\Theta(\theta_0)$. We see that for dependent indicators I_j we obtain a positive value for the probability of ruin even when the initial capital is infinitely large. Of course, when $F_\Theta(\theta_0) > 0$, e.g. if the induced random variable Θ has support $[0, \infty)$.

If the claims X_i have a geometric distribution with parameter β, then the unconditional probability of ruin is equal to:

$$\psi(u) = \frac{\beta^{u+1}}{1-\beta} \int_{\theta_0}^{\infty} \frac{q(\theta)}{(1-q(\theta))^{u+1}} dF_\Theta(\theta) + F_\Theta(\theta_0) \,,$$

where the limiting value is given by $\theta_0 = -\dfrac{\ln(1-\beta)}{g(q)}$.

The random variable Θ induced by the Archimedean copula can be treated as an external factor affecting all the binary random variables I_j. For instance, it may describe the impact of macroeconomic factors: crises, changes in the prices of raw materials or inflation; climatic factors: floods, fires, earthquakes, volcano eruptions; or political factors: wars or government crises.

Independent random variables correspond to the copula

$$\Pi(u_1, \ldots, u_n) = u_1 \cdot \ldots \cdot u_n$$

and the copula

$$M(u_1, \ldots, u_n) = \min(u_1, \ldots, u_n)$$

generates the second extreme case – strict dependence, also called comonotonicity. For every copula, we obtain the following inequality

$$C(u_1, \ldots, u_n) \le M(u_1, \ldots, u_n).$$

In practice, families of Archimedean copulas characterized by a parameter are often used. The parameter reflects the degree of dependence and its value is strictly connected with the value of the Kendall or Spearman coefficient of rank correlation [7], [6]. For $n > 2$, every Archimedean copula C satisfies the inequality:

$$\Pi(u_1, \ldots, u_n) \le C(u_1, \ldots, u_n).$$

Now, we present commonly used families of copulas.

a) Clayton family:

$$C_\alpha(u_1, \ldots, u_n) = (u_1^{-\alpha} + \ldots + u_n^{-\alpha} - n + 1)^{-1/\alpha} \,,$$

for $\alpha > 0$, with the generator $g(u) = u^{-\alpha} - 1$. The case in the limit as $\alpha = 0$ corresponds to independence and $\alpha = \infty$ implies strict dependence. The induced random variable Θ has a gamma distribution $\mathrm{Ga}\left(\frac{1}{\alpha}, 1\right)$. Thus the conditional probability of the appearance of a claim takes the form

$$q_\alpha(\theta) = e^{\theta(1-q^{-\alpha})},$$

and the threshold value θ_0, denoted by θ_α, is a decreasing function of α:

$$\theta_\alpha = \frac{q^\alpha \ln m}{1-q^\alpha}.$$

b) Frank family:

$$C_\alpha(u_1, ..., u_n) = -\frac{1}{\alpha}\ln\left(1 + \frac{(e^{-\alpha u_1} - 1)\ldots(e^{-\alpha u_n} - 1)}{(e^{-\alpha} - 1)^{n-1}}\right),$$

for $0 \le \alpha$ with the generator $g(u) = -\ln\frac{\exp(-\alpha u) - 1}{\exp(-\alpha) - 1}$. The case $\alpha = 0$ corresponds to independence and $\alpha = \infty$ implies strict dependence. The induced random variable Θ is a discrete random variable and has a logarithmic distribution. The threshold value θ_α is equal to

$$\theta_\alpha = \frac{\ln m}{\ln(\exp(-\alpha) - 1) - \ln(\exp(-\alpha q) - 1)}.$$

c) Gumbel family:

$$C_\alpha(u_1, ..., u_n) = \exp(-((-\ln u_1)^\alpha + ... + (-\ln u_n)^\alpha)^{1/\alpha}),$$

for $\alpha \ge 1$, with the generator $g(u) = (-\ln u)^\alpha$. The cases $\alpha = 1$ and $\alpha = \infty$ correspond to independence and strict dependence, respectively. The random variable Θ has an α-stable distribution and the threshold value θ_α takes the form

$$\theta_\alpha = \frac{\ln m}{(-\ln q)^\alpha}.$$

Example 2. Let us study the case where claims appear with probability $q = 0.3$, they have a geometric distribution with parameter $\beta = 0.5$ and the dependence structure of indicators I_i is described by the Clayton copula C_α. The value of this parameter

α reflects the degree of dependence: $\tau = \alpha/(a + 2)$, where τ is the Kendall coefficient of correlation. Let us assume, that the parameter α takes in turn the values: 0, 0.1, 1, 2, 4 and ∞, i.e. the values of the Kendall coefficient equal: 0, 0.048, 1/3, 1/2, 2/3 and 1, respectively. Table 1 gives the probabilities of ruin for these values of the parameter α reflecting the degree of dependence between the I_j. The probabilities of ruin for different values of initial capital are presented in fig. 2.

Table 1. Probabilities of ruin for selected values of the parameter α

		α				
u	independence	0.1	1	2	4	Strict dependence
0	0.42857	0.45766	0.42055	0.38448	0.35064	0.3
1	0.30612	0.36106	0.38117	0.36161	0.33939	0.3
2	0.21866	0.29190	0.35435	0.34607	0.33170	0.3
3	0.15618	0.24166	0.33557	0.33521	0.32629	0.3
4	0.11156	0.20462	0.32206	0.32739	0.32236	0.3
5	0.07969	0.17689	0.31208	0.32161	0.31944	0.3
6	0.05692	0.15583	0.30452	0.31722	0.31721	0.3
7	0.04066	0.13959	0.29866	0.31380	0.31546	0.3
8	0.02904	0.12689	0.29401	0.31109	0.31407	0.3
9	0.02074	0.11682	0.29026	0.30889	0.31293	0.3
10	0.01482	0.10872	0.28718	0.30708	0.31199	0.3
15	0.00275	0.08515	0.27755	0.30137	0.30900	0.3
20	0.00051	0.07443	0.27255	0.29837	0.30741	0.3
25	0.00010	0.06853	0.26951	0.29652	0.30643	0.3
30	0.00002	0.06486	0.26746	0.29528	0.30576	0.3
∞	0.00000	0.04961	0.25700	0.28882	0.30227	0.3

Source: Author's own work.

We see that for small values of initial capital u, not greater than 6, there is no regularity. We observe different orderings of the values of the probability of ruin with respect to the values of the parameter α, i.e. the degree of dependence, for different values of u. This situation also takes place for larger values of initial capital u (see fig. 3). We observe a rapid growth in the probability of ruin for small values of the parameter α and thereafter a slow decrease to the limiting value, 0.3. We obtain the highest value of the probability of ruin, equal to 0.30435, for α = 7.064. This corresponds to a value of τ = 0.779 for the Kendall coefficient of correlation.

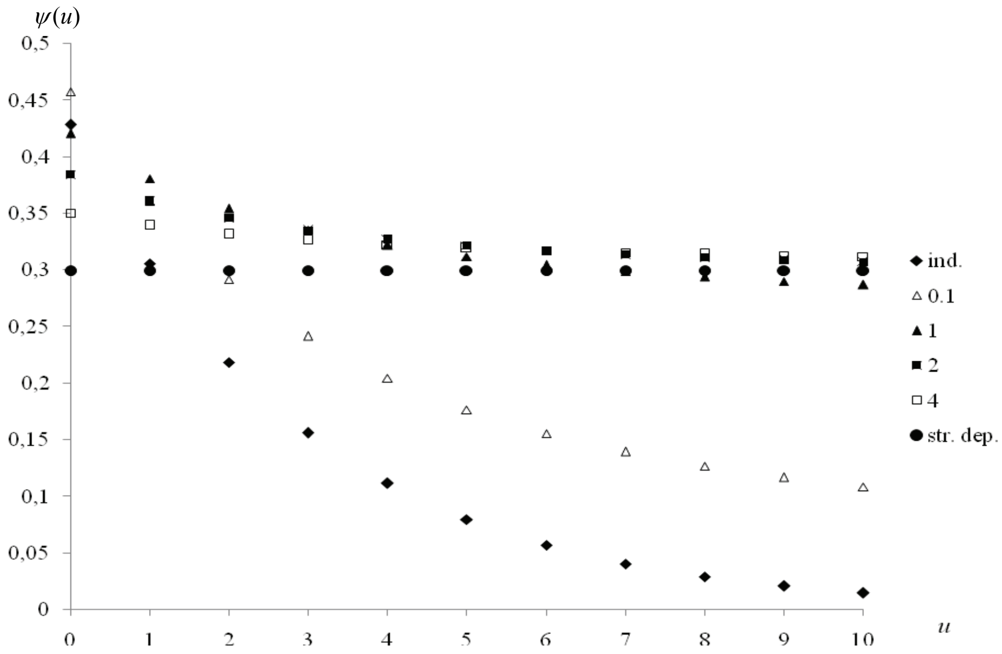

Fig. 2. The probabilities of ruin for selected values of the parameter α
Source: Author's own work.

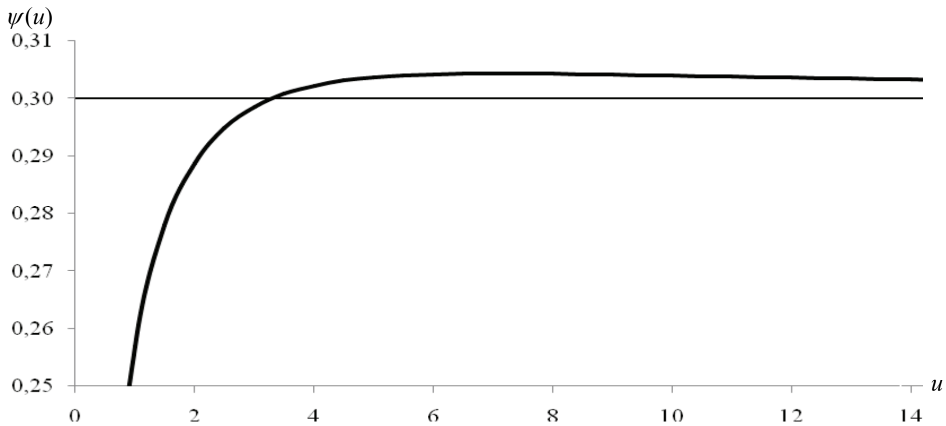

Fig. 3. Probability of ruin for an infinitely large initial capital u for different values of α
Source: Author's own work.

6. Markov binomial model

Let us assume that the dependence of the binary random variables I_j is described by a Markov chain. In this case, the indicators I_0, I_1, I_2, ... create a stationary Markov chain with binary state space {0, 1}, with matrix of transition probabilities:

$$\mathbf{P} = \begin{pmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{pmatrix} = \begin{pmatrix} p+\pi q & q-\pi q \\ p-\pi p & q+\pi p \end{pmatrix},$$

where $p_{ij} = P(I_{k+1} = j | I_k = i)$ for $i, j \in \{0, 1\}$, $0 < p < 1$, $q = 1 - p$, $0 \leq \pi \leq 1$ and with initial probabilities [1, 2]

$$P(I_0 = 0) = p, \qquad P(I_0 = 1) = q.$$

The stationarity of the Markov chain implies, that $P(I_k = 1) = q$, for all $k \geq 1$.

The parameter π characterizing the transition matrix **P** reflects the degree of dependence between the random variables in the Markov chain. The coefficient of correlation for the pair of random variables I_k and I_{k+h} is equal to [1]

$$\rho\,(I_k, I_{k+h}) = \pi^h.$$

In other words, the parameter π is the coefficient of correlation for neighbouring indicators. We obtain the classical case of independence for $\pi = 0$. In this case, the transition matrix takes the following form:

$$\mathbf{P} = \begin{pmatrix} p & q \\ p & q \end{pmatrix}.$$

When $\pi = 1$, we obtain strict dependence with the transition matrix **P** being the identity matrix.

Under these assumptions, we can determine the conditional probability of ruin, which is dependent on the initial state of the Markovian process. This probability is described by the formula

$$\psi(u|i) = P(U(t) < 0 \text{ for some } t \mid U(0) = u, I_0 = i),$$

where $i = 0, 1$. Thus the unconditional probability of ruin, $\psi(u)$, is equal to [1, 2]

$$\psi(u) = p\,\psi(u|0) + q\,\psi(u|1).$$

We can determine these probabilities for $0 \leq \pi < 1$ using the recursion formulas [1, 2]:

$$\psi_\pi(0\,|\,0) = \frac{q}{p}(m-1)\,, \tag{1}$$

$$\psi_\pi(0\,|\,1)=\frac{\pi(1-f(1))+p_{10}\psi(0\,|\,0)}{p_{00}-\pi f(1)} \tag{2}$$

and for $u = 1, 2, ...$

$$\psi_\pi(u\,|\,0)=\psi_\pi(0\,|\,0)-\frac{q}{p}\sum_{k=1}^{u}(1-F(k))(1-\psi_\pi(u-k\,|\,1)), \tag{3}$$

$$\psi_\pi(u\,|\,1)=\psi_\pi(0\,|\,1)-\sum_{k=1}^{u}\frac{\pi f(k+1)+p_{01}(1-F(k))}{p_{00}-\pi f(1)}(1-\psi_\pi(u-k\,|\,1)). \tag{4}$$

We obtain the case of strict dependence studied in point 4 when $\pi = 1$. In this case,

$$\psi_c(u\,|\,0)=0, \tag{5}$$

$$\psi_c(u\,|\,1)=\begin{cases}1 & \text{for } m>1\\ 0 & \text{for } m=1\end{cases} \tag{6}$$

and

$$\psi_c(u)=\begin{cases}q & \text{for } m>1\\ 0 & \text{for } m=1.\end{cases}$$

Now, we call attention to the fact that formulas (1), (2), (3) and (4) apply in the situation where the parameter $\pi < 1$. We obtain the limiting conditional probability of ruin ψ_g when $\pi = 1$, which is different than probability done by formulas (5) and (6) (see Example 3). For $m > 1$:

$$\psi_g(u\,|\,0)=\frac{q}{p}(m-1),$$

$$\psi_g(u\,|\,1)=1$$

and

$$\psi_g(u)=gm,$$

for $u = 0, 1, 2, ...$ We see that the limiting probability of ruin $\psi_g(u)$, i.e. when $\pi = 1$, is always greater than the probability of ruin for strictly dependent claims. When claims are deterministic and equal to 1, then the probability of ruin equals zero in both cases. The authors showed in [1] that the unconditional probability of ruin increases with the degree of dependence, i.e.

$$\psi_{\pi_1}(u)<\psi_{\pi_2}(u),$$

when $\pi_1 < \pi_2 < 1$.

We can present the exact form of the probability of ruin in some situations, as in the case of independent claims [2]. When claims X take one of two values: 1 and 2 only, we have

$$\psi(u\,|\,i)=\psi(0\,|\,i)\left(\frac{p_{11}f(2)}{p_{00}-\pi f(1)}\right)^{u}=\psi(0\,|\,i)\left(\frac{q+\pi p}{p+\pi(f(2)-p)}f(2)\right)^{u},$$

and

$$\psi(u)=\psi(0)\left(\frac{p_{11}f(2)}{p_{00}-\pi f(1)}\right)^{u},$$

where $i = 0, 1$, $\psi(0\,|\,0)=\dfrac{q}{p}f(2)$, $\psi(0\,|\,1)=\dfrac{q+\pi p}{p+\pi(f(2)-p)}f(2)$ and $\psi(0)=\dfrac{1+\pi f(2)}{p+\pi(f(2)-p)}qf(2)$

If the claims are deterministic and equal to 2, then we obtain the following expressions from the above formulas

$$\psi(u\,|\,i)=\psi(0\,|\,i)\left(\frac{p_{11}}{p_{00}}\right)^{u}=\psi(0\,|\,i)\left(\frac{q+\pi p}{p+\pi q}\right)^{u},$$

$$\psi(u)=\psi(0)\left(\frac{p_{11}}{p_{00}}\right)^{u},$$

where $\psi(0\,|\,0)=\dfrac{q}{p}$, $\psi(0\,|\,1)=\dfrac{q+\pi p}{p+\pi q}$, and $\psi(0)=\dfrac{q+\pi q}{p+\pi q}$.

When the claims X have a geometric distribution: $f(k) = (1-\beta)\beta^{k-1}$, then the conditional probabilities of ruin take the form:

$$\psi(u\,|\,i)=\psi(0\,|\,i)\left(\frac{\beta}{p_{00}-\pi(1-\beta)}\right)^{u},$$

and the unconditional probabilities are equal to

$$\psi(u)=\psi(0)\left(\frac{\beta}{p_{00}-\pi(1-\beta)}\right)^{u},$$

where

$$i = 0, 1,\ \psi(0\,|\,0) = \frac{\beta q}{p(1-\beta)},$$

$$\psi(0\,|\,1) = \frac{q + \pi(p-\beta)}{p + \pi(\beta - p)},$$

and $\psi(0) = \dfrac{\beta q}{(1-\beta)(p + \pi(\beta - p))}$.

Example 3. We study the impact of the degree of dependence, measured by the parameter π on the probability of ruin in the case where the claims have a geometric distribution. Let $\beta = 0.6$ and $q = 0.3$. Thus the expected value of a claim equals $m = 2.5$. The values of the probabilities of ruin for different values of the initial capital u and the parameter π are contained in table 2. While figure 3 presents the graphs of such probabilities for $\pi = 0, 0.2, 0.5, 0.8, 1$ and the limit as $\pi \to 1$.

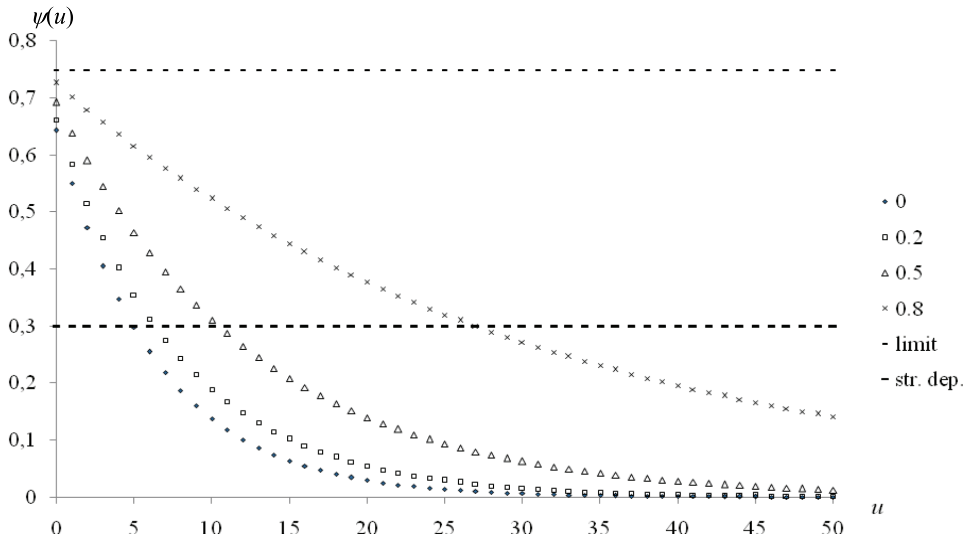

Fig. 4. Values of the probabilities of ruin for different values of parameter π
Source: Author's own work.

The limiting probability of ruin is equal to $\psi_q(u) = 0.75$ and we obtain $\psi_c(u) = 0.3$ for the case of strict dependence. We see that the probability of ruin increases with the degree of dependence, represented by the parameter π. This fact agrees with previous investigations. This probability of ruin is greater than the probability of ruin for strictly dependent claims with small values of initial capital u and for $\pi < 1$, e.g. for $u < 7$ and $\pi = 0.2$. While we obtain a reversed dependency for greater values of initial capital u.

Table 2. The values of probabilities of ruin for different values of initial capital u and parameter π

u	π				u	π			
	0	0.2	0.5	0.8		0	0.2	0.5	0.8
0	0.64286	0.66176	0.69231	0.72581	20	0.02946	0.05414	0.13966	0.37671
1	0.55102	0.58391	0.63905	0.70239	25	0.01363	0.02896	0.09359	0.31975
2	0.47230	0.51521	0.58990	0.67974	30	0.00631	0.01549	0.06272	0.27140
3	0.40483	0.45460	0.54452	0.65781	35	0.00292	0.00828	0.04204	0.23036
4	0.34700	0.40112	0.50263	0.63659	40	0.00135	0.00443	0.02817	0.19553
5	0.29743	0.35393	0.46397	0.61605	45	0.00062	0.00237	0.01888	0.16596
6	0.25494	0.31229	0.42828	0.59618	50	0.00029	0.00127	0.01265	0.14086
7	0.21852	0.27555	0.39533	0.57695	60	6.18E-05	0.00036	0.00568	0.10148
8	0.18730	0.24313	0.36492	0.55834	70	1.32E-05	0.00010	0.00255	0.07311
9	0.16054	0.21453	0.33685	0.54033	80	2.83E-06	2.97E-05	0.00115	0.05267
10	0.13761	0.18929	0.31094	0.52290	90	6.07E-07	8.48E-06	0.00051	0.03795
15	0.06367	0.10124	0.20839	0.44383	100	1.3E-07	2.43E-06	0.00023	0.02734

Source: Author's own work.

7. Conclusion

Discrete risk processes with dependent moments of the appearance of claims have been investigated in this paper. This is a generalization of classical risk processes based on the assumption of the independence of random variables or random processes appearing in them. Three versions of dependent risk processes have been presented. Strict dependence is assumed in the first version. The second version is based on Archimedean copulas. In the third version, the random variables describing the moments of the appearance of claims create a Markov chain. In each case, the probability of ruin and the influence of the degree of dependence on this probability have been studied. Only in the third case can we observe a consistent effect. In this case, the probability of ruin increases with the degree of dependence for all values of initial capital. There is no such regularity in the case of Archimedean copulas. This influence depends on the value of initial capital.

We can also investigate discrete risk processes with dependent values of claims or processes characterizing dependencies between the times at which claims occur and their value. These processes will be the subject matter of future papers by the author.

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