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A SIMPLE PROOF OF THE FOUR-COLORS THEOREM

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Abstract: The paper includes an inductive proof of the four-color theorem. The notions of producing maps and minimum local colorings are applied.

Keywords: minimum local coloring, extension of local coloring, dissecting ring, simple map, production of maps.

JEL Classification: C65, C78, C62.

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The paper deals with typical planar maps or equivalent spherical maps. A map implies vertices, borders, countries; a vertex is a point where borders meet, a border is a line which demarcates countries, a country is a polygon with edges which can be curved. Countries sharing a vertex V are called a local map. It follows from the four-color theorem that there exists a consistent local coloring with three or less colors in each vertex; this is called a local coloring in vertex V. This coloring is obviously consistent, i.e. neighboring countries sharing a common border are colored with different colors. Countries sharing only a common vertex are not regarded as adjacent. A map M is a product of maps M_1 and M_2, if it includes a dissecting belt that is composed of three or less countries (Figures 1 and 6).

A ring separating maps M_1 and M_2 belongs both to M_1 and M_2 – it connects them. The map M_1 is represented by three dots on the outside of the dissecting belt, while the map M_2 – three dots inside this belt (Figures 2 and 3).

An exterior circle in figures is always contracted to one point – the north pole that is added by the one-point compactification of a plane – obviously, it is a sphere; the point can be any point or a vertex of the map if the borders contact the exterior circle.

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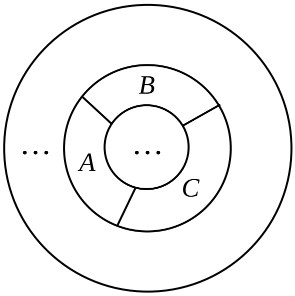

Fig. 1. A ring dissecting a map

Source: own elaborations.

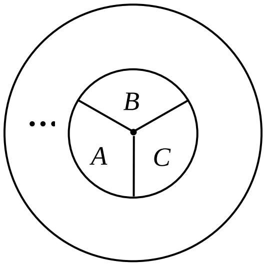

Fig. 2. Map M_1

Source: own elaborations.

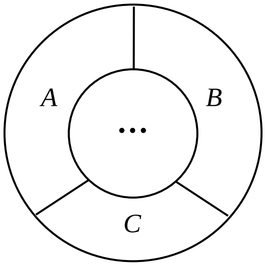

Fig. 3. Map M_2

Source: own elaborations.

Remark. The borders of a belt dissecting countries which belong to it can be reduced to one point in particular. Thus, there can be three more types of dissecting belts which will be used in our proof; they are presented in Figure 4. Dissecting belts in which neighboring countries share one common point will be regarded as special dissections.

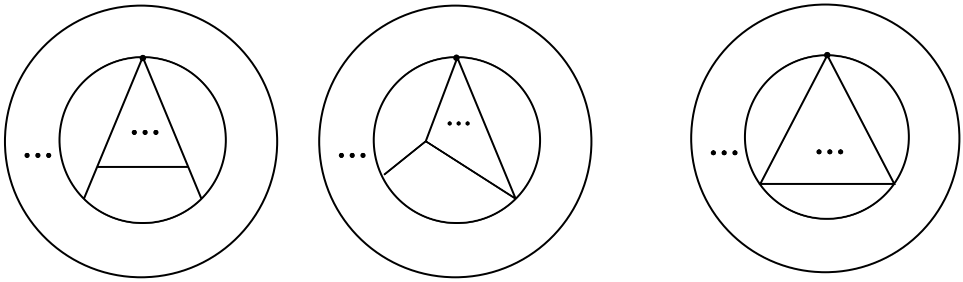

Fig. 4. Special belts

Source: own elaborations.

Certainly a map can contain dissecting belts made of two countries, and even of just one. Dissecting belts reduce the coloring of complex maps to the coloring of smaller maps, and then to producing a combined map.

Lemma. *If a map is simple, i.e. it is not a product of smaller maps, then there is a local coloring with two colors in each vertex of the map when a vertex is even – joins an even number of countries, or with three colors – when a vertex is odd.*

Local coloring is called minimum coloring if only two or three colors are used, but each one only once. If we deal with minimum coloring and two colors, then any border ending at vertex V is reduced to this vertex; if three colors were used, then we reduce a border adjacent to a country colored with the third color, that was used only once. After this operation, there are several countries in vertex V which are not colored. Since a map M is indivisible, an initial minimum coloring can be extended to a minimum coloring of map M' which is obtained from M by reducing one border. Naturally, there is a minimum coloring in each vertex of a simple map.

Theorem. *If a map is simple, then each minimum coloring extends to a consistent coloring of the whole map with four or fewer colors.*

The proof is inductive. If a map includes n or less borders, then the theorem holds. Let M have $n + 1$ borders. We consider any minimum coloring in the vertex V. By inductive assumption, when a reduced map is simple, there exists an extension of a larger minimum coloring to a consistent coloring with four or less colors. If a reduced map M' is not simple, then it is a product of smaller maps. We apply induction to smaller maps or divide them further into smaller factors. A dissecting belt *M*' always passes through vertex *V* and is shaped as shown in Figure 4, which certainly is a special belt. If countries *A* and *B* belonging to a minimum coloring of *V* and also to a dissecting belt have different colors, then maps M_1 and M_2 are as shown in Figures 2 and 3. If these countries have a consistent color, then we combine them into one country and obtain a belt made of only two countries, as shown in Figure 5. One can always combine maps M_1 and M_2 without spoiling the minimum coloring of map *M*' obtained from *M* by the reduction of one border which ends at point *V*. Next, the restoration of the reduced border brings back map *M* and we obtain an extension of minimum coloring to the consistent coloring of the whole map *M* with four or less colors. Thus the proof is completed.

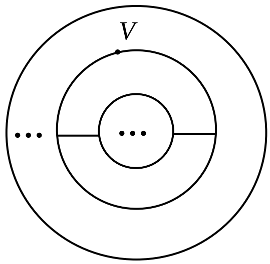

Fig. 5. One color on a dissecting path

Source: own elaborations.

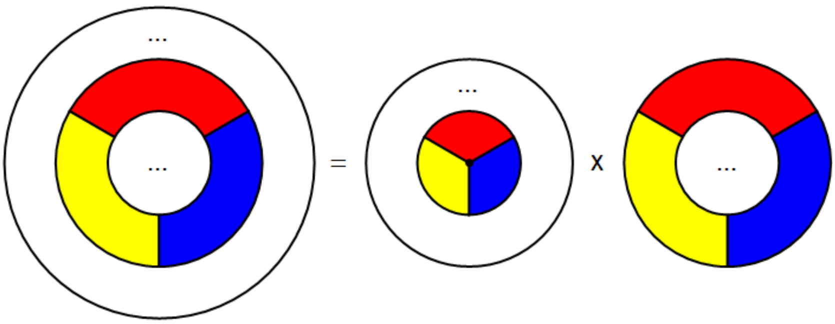

Fig. 6. Product of maps

Source: own elaborations.

Note. The literature on the subject is rich, different and easy to get to. The lack of references means that the author did not draw on it. Any possible similarities arise from the logic of the object.