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APPLICATION OF THE REPRESENTATIONS OF SYMMETRIC GROUPS TO CHARACTERIZE SOLUTIONS OF GAMES IN PARTITION FUNCTION FORM

A different perspective from the more "traditional" approaches to studying solutions of games in partition function form has been presented. We provide a decomposition of the space of such games under the action of the symmetric group, for the cases with three and four players. In particular, we identify all the irreducible subspaces that are relevant to the study of linear symmetric solutions. We then use such a decomposition to derive a characterization of the class of linear and symmetric solutions, as well as of the class of linear, symmetric and efficient solutions.

Keywords: *games in partition function form, value, representation theory, symmetric group*

1. Introduction

The problem of distributing the surplus generated by a collection of people who are willing to cooperate with one another is well captured by cooperative game theory. It is assumed that a game is characterized by giving the value of each possible coalition from a set of players. Several models describe the value of a coalition by means of a real valued characteristic function, which is defined on the set of all subsets of the set of players. However, in the case of an economy with externalities, one cannot easily recommend a distribution of the joint gains, as it depends on the organizational structure which has been formed. In this context, Lucas and Thrall [8]

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introduced a new formulation for the theory of cooperative games in terms of partition functions. They assumed that players divide into coalitions, forming a partition of the set of players. According to this model, a partition function assigns a value to each pair consisting of a coalition and a partition which includes that coalition. The advantage of this model is that it takes into account both internal factors (a coalition itself) and external factors (the coalition structure) that may affect cooperation outcomes and allows us to analyze cooperation problems more deeply.

There have been many papers dealing with solutions of games in partition function form. The first author that proposed a concept for the value of this type of game was Myerson [10], and then Bolger [20] derived a class of linear, symmetric and efficient values for games in partition function form. More recently, Albizuri et al. [1], Macho-Stadler [9], Ju [7], Pham Do and Norde [11], and De Clippel and Serrano [3] apply an axiomatic approach to find a value.

In this paper, linear symmetric solutions of games in partition function form have been studied for the cases of three and four players with the innovative use of basic representation theory to describe the group of permutations of the set of players presenting a different perspective from the more "traditional" approaches.

Very roughly speaking, representation theory is a general tool for studying abstract algebraic structures by representing their elements as linear transformations of vector spaces. This is useful, since every permutation may be thought of as a linear map[2] which presents the information in a more clear and concise way. It is a beautiful mathematical subject which has many applications, ranging from the number theory and combinatorics to geometry, probability theory, quantum mechanics and quantum field theory. It was recently used by Hernández-Lamoneda et al. [5] to study solutions of games in characteristic function form, where they propose representation theory as a natural tool for research in cooperative game theory.

Briefly, what we do is to derive a direct sum decomposition of the space of games in partition function form and the space of payoffs into "elementary pieces". According to this decomposition, any linear symmetric solution, when restricted to any such elementary piece, is either zero or a multiple of a single scalar. Therefore, all linear symmetric solutions may be written as a sum of trivial maps.

Having a global description of all linear and symmetric solutions, it is easy to understand the restrictions imposed by the efficiency axiom. We then use such a decomposition to provide, in a very economical way, a characterization of the class of linear symmetric solutions and a general expression for all linear, symmetric and efficient solutions.

The paper is organized as follows. In the next section, we first recall the main basic features of games in partition function form. A decomposition of the space of

[2] A precise definition will be provided in Sec. 3.

three player games in partition function form is introduced in Section 3. In the same section, we show an application of this decomposition by giving characterizations of linear symmetric solutions. In Section 4, we discuss the decomposition for the case of four player games and Section 5 concludes the paper. Long proofs are presented in the Appendix.

To finish this introduction, we give a comment on the methods employed in this paper. Although it is true that the characterization results could be proved without any explicit mention of basic representation theory with regard to symmetric groups, we feel that by doing that we would be withholding valuable information from the reader. This algebraic tool, we believe, sheds new light on the structure of the space of games in partition function form and their solutions. Part of the purpose of the present paper is to share this viewpoint with the reader.

To make the paper as self contained as possible, we have included an Appendix with some facts we need regarding basic representation theory.

2. Framework and notation

In this section, we give some concepts and notation related to n-person games in partition function form, as well as a brief subsection containing preliminaries related to integer partitions, since they are a key subject in subsequent derivations.

2.1. Games in partition function form

Let $N = \{1, 2, ..., n\}$ be a fixed nonempty finite set, and let the members of N be interpreted as players in some game. Given 0N, let CL be the set of all coalitions (nonempty subsets) of N, $CL = \{S \mid S \subseteq N, S \neq \varnothing\} = 2^N \setminus \{\varnothing\}$. Let PT be the set of partitions of N, so

$$\{S_1, S_2, ..., S_m\} \in PT \text{ iff } \bigcup_{i=1}^{m} S_i = N, \qquad S_j \neq \varnothing \ \forall j, \qquad S_j \cap S_k = \varnothing \forall j \neq k$$

Also, let $ECL = \{(S, Q) \mid S \in Q \in PT\}$ be the set of embedded coalitions, that is the set of coalitions together with specifications as to how the other players are aligned.

For the sake of concision, we often denote by SQ the embedded coalition (S, Q), and omit braces and commas in the description of subsets (for example: $12\{12, 3\}$

instead of ({1, 2}, {{1, 2}, {3}}). Additionally, we will denote the cardinality of a set by its corresponding lower-case letter, for instance $n = |N|,\ s = |S|,\ q = |Q|$.

Definition 1. A mapping

$$w : ECL \to R$$

that assigns a real value, $w(S,Q)$, to each embedded coalition (S,Q) is called a game in partition function form. The set of games in partition function form with player set N is denoted by G, i.e.,

$$G = G^{(n)} = \{w \mid w : ECL \to R\}$$

The value $w(S,Q)$ represents the payoff of coalition S, given that the coalition structure Q forms. In this kind of game, the value of some coalition depends not only on what the players of such a coalition can jointly obtain, but also on the way in which the other players are organized. We assume that, in any game situation, the universal coalition N (embedded in $\{N\}$) will actually form, so that the players will have $w(N,\{N\})$ to divide among themselves. But we also anticipate that the actual allocation of this value will depend on all the other potential values $w(S, Q)$, as they influence the relative bargaining strengths of the players.

Given $w_1, w_2 \in G$ and $c \in R$, we define the sum $w_1 + w_2$ and the product cw_1, in G, in the usual way, i.e.

$$(w_1 + w_2)(S,Q) = w_1(S,Q) + w_2(S,Q) \qquad \text{and} \qquad (cw_1)(S,Q) = cw_1(S,Q)$$

respectively. It is easy to verify that with these operations G is a vector space.

A solution is a function $\varphi : G \to R^n$. If φ is a solution and $w \in G$, then we can interpret $\varphi_i(w)$ as the payoff which player i should expect from the game w.

Now, the group of permutations of N, $S_n = \{\theta : N \to N \mid \theta \text{ is bijective}\}$, acts on CL and on ECL in the natural way; i.e., for $\theta \in S_n$:

$$\theta(S) = \{\theta(i) \mid i \in S\}$$

$$\theta(S_1, \{S_1, S_2, ..., S_l\}) = (\theta(S_1), \{\theta(S_1), \theta(S_2), ..., \theta(S_l)\})$$

Also, S_n acts on the space of payoff vectors, R^n:

$$\theta(x_1, x_2, ..., x_n) = (x_{\theta(1)}, x_{\theta(2)}, ..., x_{\theta(n)})$$

Next, we define the usual linearity, symmetry and efficiency axioms, which solutions are required to satisfy in the framework of cooperative game theory.

Axiom 1. Linearity. The solution φ is linear if $\varphi(w_1 + w_2) = \varphi(w_1) + \varphi(w_2)$ and $\varphi(cw_1) = c\varphi(w_1)$, for all $w_1, w_2 \in G$ and $c \in R$.

Axiom 2. Symmetry. The solution φ is said to be symmetric if and only if $\varphi(\theta w) = \theta\varphi(w)$ for every $\theta \in S_n$ and $w \in G$, where the game θw is defined as

$$(\theta w)(S,Q) = w[\theta^{-1}(S,Q)]$$

Axiom 3. Efficiency. The solution φ is efficient if $\sum_{i\in N} \varphi_i(w) = w(N,\{N\})$ for all $w \in G$.

Myerson [10] proceeds axiomatically and proposes a value that extends the well known Shapley value [12] which is defined for TU games. His proposal satisfies the axioms of linearity, symmetry, efficiency and the "null" player property that states that players who have no effect on the outcome should neither receive nor pay anything.

The Myerson value of a player is given by

$$\psi_i(w) = \sum_{(S,Q)\in ECL} (-1)^{q-1}(q-1)!\left(\frac{1}{n} - \sum_{T\in Q\setminus\{S\}, i\notin T} \frac{1}{(q-1)(n-t)}\right) w(S,Q)$$

2.2. Integer partitions

A partition of a nonnegative integer is a way of expressing it as an unordered sum of other positive integers, and it is often written in tuple notation. Formally:

Definition 2. $\lambda = [\lambda_1, \lambda_2, ..., \lambda_l]$ is a partition of *n* iff $\lambda_1, \lambda_2, ..., \lambda_l$ are positive integers and $\lambda_1 + \lambda_2 + \cdots + \lambda_l = n$. Two partitions which only differ in the order of their elements are considered to be the same partition.

The set of all partitions of *n* will be denoted by $\Pi(n)$, and, if $\lambda \in \Pi(n)$, $|\lambda|$ is the number of elements of λ.

For example, the partitions of $n = 4$ are $[1,1,1,1]$, $[2,1,1]$, $[2,2]$, $[3,1]$, and $[4]$. Sometimes we will abbreviate this notation by dropping the commas, so $[2,1,1]$ becomes $[211]$.

If $Q \in PT$, there is a unique partition $\lambda_Q \in \Pi(n)$, associated with Q, where the elements of λ_Q are exactly the cardinalities of the elements of Q. In other words, if $Q = \{S_1, S_2, ..., S_m\} \in PT$, then $\lambda_Q = [s_1, s_2, ..., s_m]$.

For a given $\lambda \in \Pi(n)$, we represent by λ° the set of numbers determined by the λ_i and for $k \in \lambda^\circ$, we denote by m_k the multiplicity of k in the partition λ. So, if $\lambda = [4, 2, 2, 1, 1, 1]$, then $\lambda^\circ = \{1, 2, 4\}$ and $m_1 = 3,\ m_2 = 2,\ m_4 = 1$.

3. Representations

Precise definitions and some proofs for this section may be found in the Appendix at the end of the paper. Nevertheless, for the sake of easier reading, we repeat a few definitions here, sometimes in a less rigorous but more accessible, manner.

The group S_n acts naturally on the space of games in partition function form, G, via linear transformations (i.e., G is a representation of S_n). That is to say, each permutation $\theta \in S_n$ corresponds to a linear, invertible transformation, which we still call θ, of the vector space G, namely

$$(\theta w)(S, Q) = w[\theta^{-1}(S, Q)]$$

for every $\theta \in S_n$, $w \in G$ and $(S, Q) \in ECL$.

Moreover, this assignment preserves multiplication (i.e., is a group homomorphism) in the sense that the linear map corresponding to the product of the two permutations $\theta_1\theta_2$ is the product (or composition) of the maps corresponding to θ_1 and θ_2, in that order.

Similarly, the space of payoff vectors, R^n, is a representation of S_n:

$$\theta(x_1, x_2, ..., x_n) = (x_{\theta(1)}, x_{\theta(2)}, ..., x_{\theta(n)})$$

Definition 3. Let X_1 and X_2 be two representations of the group S_n. A linear map $T : X_1 \to X_2$ is said to be S_n equivariant if $T(\theta x) = \theta T(x)$, for every $\theta \in S_n$ and every $x \in X_1$.

Remark 1. Notice that, in the language of representation theory, what we call a linear symmetric solution is a linear map $\varphi : G \to R^n$ that is S_n equivariant.

3.1. Decomposition of $G^{(3)}$

Definition 4. Let Y be a subspace of the vector space X.

• Y is invariant (with respect to the action of S_n) if for every $y \in Y$ and every $\theta \in S_n$, we have

$$\theta y \in Y$$

• Y is irreducible if Y itself has no invariant subspaces other than $\{0\}$ and Y itself.

We begin with the decomposition of R^n into irreducible representations which is easier, and then proceed to do the same thing for G. That is to say, we wish to write R^n as a direct sum of subspaces, each invariant with respect to all permutations in S_n, in such way that the summands cannot be further decomposed (i.e., they are irreducible).

For this, set $\mathbf{1} = (1, 1, ..., 1) \in R^n$ and

$$U_n = \langle \mathbf{1} \rangle \qquad \text{and} \qquad V_n = \{ z \in R^n \mid z \cdot \mathbf{1} = 0 \}$$

The spaces U_n and V_n are usually called the "trivial" and "standard" representations, respectively. Notice that U_n is a trivial subspace in the sense that every permutation acts like the identity transformation.

Every permutation fixes each element of U_n, so, in particular, it is an invariant subspace of R^n. Being one dimensional, it is automatically irreducible. Its orthogonal complement, U_n, consists of all vectors such that the sum of their coordinates is zero. Clearly, if we permute the coordinates of any such vector, its sum will still be zero. Hence, V_n is also an invariant subspace.

Proposition 1. The decomposition of R^n under S_n into irreducible subspaces is:

$$R^n = U_n \oplus V_n$$

Proof. First, it is clear that $U_n \cap V_n = \{0\}$[3]. We now prove that $R^n = U_n + V_n$.

1. If $z \in (U_n + V_n)$, then $z \in R^n$, since $(U_n + V_n)$ is a subspace of R^n.

[3]Here, $\mathbf{0} = \{0, 0, \ldots, 0 \in R^n)$

2. For $z \in R^n$, let $\overline{z} = \frac{1}{n}\sum_{i=1}^{n} z_i$. Note that z can be written as

$$z = (\overline{z}, \overline{z}, \ldots, \overline{z}) + (z_1 - \overline{z}, z_2 - \overline{z}, \ldots, z_n - \overline{z})$$

and so,

$$z \in (U_n + V_n)$$

Finally, since U_n is one dimensional, then it is irreducible. To check that V_n is also irreducible, an induction argument that can be found in Hernández-Lamoneda et al. [5] may be used.

Thus, this result tells us that R^n, as a vector space with group of symmetry S_n, can be written as an orthogonal sum of the subspaces U_n and V_n, which are invariant under permutations and cannot be further decomposed.

The decomposition of G is carried out in three steps. For a given $\lambda \in \Pi(n)$, let $Q_\lambda = \{Q \in PT \mid \lambda_Q = \lambda\}$. For each $\lambda \in \Pi(n)$, define the subspace of games

$$G_\lambda = \{w \in G \mid w(S, Q) = 0, \text{ if } Q \notin Q_\lambda\}$$

Thus,

$$G = \bigoplus_{\lambda \in \Pi(n)} G_\lambda$$

whereas, for $k \in \lambda^\circ$, define the subspace

$$G_\lambda^k = \{w \in G_\lambda \mid w(S, Q) = 0, \text{ if } |S| \neq k\}$$

Then each G_λ has a decomposition $G_\lambda = \bigoplus_{k \in \lambda^\circ} G_\lambda^k$ and so we obtain the following decomposition of G:

$$G = \bigoplus_{\lambda \in \Pi(n)} \bigoplus_{k \in \lambda^\circ} G_\lambda^k = \bigoplus_{\substack{\lambda \in \Pi(n) \\ k \in \lambda^\circ}} G_\lambda^k \tag{1}$$

Each subspace G_λ^k is invariant under S_n and the decomposition is orthogonal with respect to the invariant inner product on G given by

$$\langle w_1, w_2 \rangle = \sum_{(S,Q)\in ECL} w_1(S,Q)w_2(S,Q)$$

Here, invariance of the inner product means that each permutation $\theta \in S_n$ is not only a linear map on G but an orthogonal map with respect to this inner product. Formally, $\langle \theta w_1, \theta w_2 \rangle = \langle w_1, w_2 \rangle$ for every $w_1, w_2 \in G$.

Example 1. For the case $N = \{1, 2, 3\}$, $\dim G^{(3)} = 10$ and it decomposes as follows:

$$G^{(3)} = G^1_{[1,1,1]} \oplus G^1_{[2,1]} \oplus G^2_{[2,1]} \oplus G^3_{[3]}$$

The next goal is to get a decomposition of each subspace of games G^k_λ into irreducible subspaces and hence obtain a decomposition of $G^{(3)}$.

The following games play an important role in describing the decomposition of the space of three player games:

$$u^k_\lambda(S,Q) = \begin{cases} 1 & \text{if } Q \in Q_\lambda,\ |S| = k \\ 0 & \text{otherwise} \end{cases}$$

Notice that $G^n_{[n]} = \mathbb{R}u^n_{[n]}$.

Also, for each $\lambda \in \Pi(n) \setminus \{[n]\}$, $k \in \lambda^\circ$ and $z \in V_n$; let $z^{\lambda,k} \in G^k_\lambda$ be given by

$$z^{\lambda,k}(S,Q) = \begin{cases} \sum_{i\in S} z_i & \text{if } Q \in Q_\lambda,\ |S| = k \\ 0 & \text{otherwise} \end{cases}$$

Definition 5. Suppose X_1 and X_2 are two representations of the group S_n, i.e., we have two vector spaces X_1 and X_2 where S_n acts using linear maps. We say that X_1 and X_2 are isomorphic, if there is a linear map between them which is 1–1 and onto and commutes with the respective S_n actions. Formally, there is an invertible linear map $T : X_1 \to X_2$ such that $T(\theta x) = \theta T(x)$ for each $\theta \in S_n$ and $x \in X_1$. We then write $X_1; X_2$.

For our purposes, X_1 will be an irreducible subspace of G and X_2 an irreducible subspace of $\mathbb{R}^n$.

Isomorphic representations are essentially "equal"; not only are they spaces of the same dimension, but the actions equivalent under some linear invertible map between them.

The next proposition provides us a decomposition of the space of three player games into irreducible subspaces.

Proposition 2. For $\lambda \in \Pi(3) \setminus \{[3]\}$,

$$G_\lambda^k = U_\lambda^k \oplus V_\lambda^k$$

where $U_\lambda^k = \langle u_\lambda^k \rangle$; U_3 and $V_\lambda^k = \{ z^{\lambda,k} \mid z \in V_3 \}$; V_3. The decomposition is orthogonal. (See proof in the Appendix).

Remark 2. From the above Proposition, it is not difficult to verify that

$$V_\lambda^k = \left\{ w \in G_\lambda^k \mid \sum_{\substack{(S,Q)\in ECL \\ |S|=k,\, Q\in Q_\lambda}} w(S,Q) = 0 \right\}$$

Proposition 2 gives us a decomposition of the space of three player games that is a key ingredient in our subsequent analysis.

Set $U_G = U^1_{[1,1,1]} \oplus U^1_{[2,1]} \oplus U^2_{[2,1]} \oplus U^3_{[3]}$. This is a subspace of games whose value on a given embedded coalition (S,Q), depends only on the cardinality of S and on the structure of Q[4]. According to Proposition 12, U_G is the largest subspace of $G^{(3)}$ where S_3 acts trivially[5]. Let $V_G = V^1_{[1,1,1]} \oplus V^1_{[2,1]} \oplus V^2_{[2,1]}$, then

$$G^{(3)} = U_G \oplus V_G$$

Thus, given a game $w \in G^{(3)}$, from the above we may decompose it as $w = u + v$, where in turn $u = \sum a_{\lambda,k} u_\lambda^k$ and $v = \sum z_{\lambda,k}^{\lambda,k}$. This decomposition is very well suited to study the image of w under any linear symmetric solution. This results from the following version of Schur's well known Lemma[6].

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[4]Such games may be thought of as a counterpart of symmetric games in TU games.

[5]i.e., $\theta w = w$ for each $\theta \in S_3$ and $w \in U_G$.

[6]See the Appendix for a precise statement of Schur's theorem.

Theorem 1 (Schur's Lemma). Any linear symmetric solution

$$\varphi : G^{(3)} = U_G \oplus V_G \to R^3 = U_3 \oplus V_3$$

satisfies

a) $\varphi(U_G) \subset U_3$,

b) $\varphi(V_G) \subset V_3$.

Moreover,

• for each $\lambda \in \Pi(3)$ and $k \in \lambda^\circ$, there is a constant $\alpha_{\lambda,k} \in R$ such that, for each $u \in U_\lambda^k$,

$$\varphi(u) = \alpha_{\lambda,k}(1,1,1) \in U_3$$

• for each $\lambda \in \Pi(3) \setminus \{[3]\}$ and $k \in \lambda^\circ$, there is a constant $\beta_{\lambda,k} \in R$ such that, for each $z^{\lambda,k} \in V_\lambda^k$,

$$\varphi(z^{\lambda,k}) = \beta_{\lambda,k} z \in V_3$$

For many purposes, it suffices to use merely the existence of the decomposition of the game $w \in G^{(3)}$, without having to worry about the precise form of each component. Nevertheless, it will be useful to compute each component. Thus we give a formula for computing them.

Proposition 3. Let $w \in G^{(3)}$. Then

$$w = \sum_{\substack{\lambda \in \Pi(3) \\ k \in \lambda^\circ}} a_{\lambda,k} u_\lambda^k + \sum_{\substack{\lambda \in \Pi(3)\setminus\{[3]\} \\ k \in \lambda^\circ}} z_{\lambda,k}^{\lambda,k} \tag{2}$$

where

1) $a_{\lambda,k}$ is the average of the values $w(S,Q)$ with $Q \in Q_\lambda$ and $|S| = k$:

$$a_{\lambda,k} = \frac{\displaystyle\sum_{\substack{(S,Q)\in ECL \\ Q \in Q_\lambda, |S|=k}} w(S,Q)}{\left|\{(S,Q) \in ECL \mid Q \in Q_\lambda, |S| = k\}\right|}$$

2) For each $\lambda \in \Pi(3) \setminus \{[3]\}$ and $k \in \lambda^{\circ}$:

$$z_{\lambda,k} = (-1)^{|\lambda|} k(|\lambda| - 1)\, \psi(w_{\lambda,k})$$

where ψ denotes Myerson's value and $w_{\lambda,k}$ is the G_{λ}^{k} component of w (i.e., $w_{\lambda,k}(S,Q) = w(S,Q)$ if $Q \in Q_{\lambda}$, $|S| = k$, and $w_{\lambda,k}(S,Q) = 0$ otherwise).

Proof. We start by computing the orthogonal projection of w onto U_G. Notice that $\{u_{\lambda}^{k}\}$ is an orthogonal basis for U_G, and that

$$\left\|u_{\lambda}^{k}\right\|^2 = \left|\left\{(S,Q) \in ECL \mid Q \in Q_{\lambda}, |S| = k\right\}\right|$$

Thus, the projection of w onto U_G is

$$\sum_{\substack{\lambda \in \Pi(3) \\ k \in \lambda^{\circ}}} \frac{\left\langle w, u_{\lambda}^{k}\right\rangle}{\left\langle u_{\lambda}^{k}, u_{\lambda}^{k}\right\rangle} u_{\lambda}^{k}$$

and thus

$$a_{\lambda,k} = \frac{\left\langle w, u_{\lambda}^{k}\right\rangle}{\left\langle u_{\lambda}^{k}, u_{\lambda}^{k}\right\rangle} = \frac{\sum_{\substack{(S,Q) \in ECL \\ Q \in Q_{\lambda}, |S| = k}} w(S,Q)}{\left|\left\{(S,Q) \in ECL \mid Q \in Q_{\lambda}, |S| = k\right\}\right|}$$

Now, for $\lambda \in \Pi(3) \setminus \{[3]\}$, let $w_{\lambda,k}$ be defined as above. It follows that

$$w_{\lambda,k} = a_{\lambda,k} u_{\lambda}^{k} + z_{\lambda,k}^{\lambda,k}$$

Applying Myerson's value, we obtain

$$\psi(w_{\lambda,k}) = a_{\lambda,k} \psi\left(u_{\lambda}^{k}\right) + \psi\left(z_{\lambda,k}^{\lambda,k}\right) = \frac{1}{(-1)^{|\lambda|} k(|\lambda| - 1)} z_{\lambda,k}$$

because $\psi\left(u_{\lambda}^{k}\right) = 0$ for $\lambda \in \Pi(3) \setminus \{[3]\}$ from the assumption of efficiency. $\psi\left(z_{\lambda,k}^{\lambda,k}\right)$ $= \beta_{\lambda,k} z_{\lambda,k}$ by Schur's Lemma, and the precise value of $\beta_{\lambda,k} = \dfrac{1}{(-1)^{|\lambda|} k(|\lambda| - 1)}$ is easy to compute.

Remark 3. The use of Myerson's value in order to compute $z_{\lambda,k}$ is a matter of personal taste. One could use one's own favorite linear symmetric solution – as long as it is non-zero on each V_λ^k – to compute them.

3.2. Applications

Now we show how to get characterizations of solutions easily by using the decomposition of a game given by (2) in conjunction with Schur's Lemma. We start by providing a characterization of all linear symmetric solutions $\varphi : G^{(3)} \to R^3$ in the following proposition:

Proposition 4. Linear symmetric solutions $\varphi : G^{(3)} \to R^3$ are of the form

$$\varphi_i(w) = \sum_{\substack{\lambda \in \Pi(3) \\ k \in \lambda^\circ}} \left[\gamma_{\lambda,k} \sum_{\substack{(S,Q) \in ECL \\ S \ni i, |S| = k \\ Q \in Q_\lambda}} w(S,Q) + \delta_{\lambda,k} \sum_{\substack{(S,Q) \in ECL \\ S \not\ni i, |S| = k \\ Q \in Q_\lambda}} w(S,Q) \right] \tag{3}$$

for some real numbers

$$\left\{ \gamma_{\lambda,k} \mid \lambda \in \Pi(3), k \in \lambda^\circ \right\} \cup \left\{ \delta_{\lambda,k} \mid \lambda \in \Pi(3) \setminus \{[3]\}, k \in \lambda^\circ \right\}$$

Proof. Let $\varphi : G^{(3)} \to R^3$ be a linear symmetric solution. By the previous proposition, $w \in G^{(3)}$ decomposes as

$$w = \sum_{\substack{\lambda \in \Pi(3) \\ k \in \lambda^\circ}} a_{\lambda,k} u_\lambda^k + \sum_{\substack{\lambda \in \Pi(3) \setminus \{[3]\} \\ k \in \lambda^\circ}} z_{\lambda,k}^{\lambda,k}$$

Without loss of generality, we take $i = 1$, then

$$\varphi_1(w) = \sum_{\substack{\lambda \in \Pi(3) \\ k \in \lambda^\circ}} a_{\lambda,k} \varphi_1\left(u_\lambda^k\right) + \sum_{\substack{\lambda \in \Pi(3) \setminus \{[3]\} \\ k \in \lambda^\circ}} \varphi_1\left(z_{\lambda,k}^{\lambda,k}\right)$$

Now, from Schur's Lemma and using Proposition 3 again, we have

$$\begin{aligned}\varphi_1(w) &= \sum_{\substack{\lambda\in\Pi(3)\\ k\in\lambda^\circ}} a_{\lambda,k}\alpha_{\lambda,k} + \sum_{\substack{\lambda\in\Pi(3)\setminus\{[3]\}\\ k\in\lambda^\circ}} \beta_{\lambda,k}\left(z_{\lambda,k}\right)_1 \\ &= \sum_{\substack{\lambda\in\Pi(3)\\ k\in\lambda^\circ}} \alpha'_{\lambda,k} \sum_{\substack{(S,Q)\in ECL\\ Q\in Q_\lambda,|S|=k}} w(S,Q) + \sum_{\substack{\lambda\in\Pi(3)\setminus\{[3]\}\\ k\in\lambda^\circ}} \beta'_{\lambda,k}\psi_1\left(w_{\lambda,k}\right)\end{aligned}$$

where

$$\alpha'_{\lambda,k} = \frac{\alpha_{\lambda,k}}{\left|\{(S,Q)\in ECL \mid Q\in Q_\lambda, |S|=k\}\right|}$$

and

$$\beta'_{\lambda,k} = \beta_{\lambda,k}(-1)^{|\lambda|}k(|\lambda|-1)$$

Notice that

$$\psi_1(w_{[111],1}) = -\frac{1}{3}w(1\{1,2,3\}) + \frac{1}{6}w(2\{1,2,3\}) + \frac{1}{6}w(3\{1,2,3\})$$

$$\psi_1(w_{[21],1}) = \frac{2}{3}w(1\{1,23\}) - \frac{1}{3}w(2\{2,13\}) - \frac{1}{3}w(3\{3,12\})$$

$$\psi_1(w_{[21],2}) = \frac{1}{6}w(12\{3,12\}) + \frac{1}{6}w(13\{2,13\}) - \frac{1}{3}w(23\{1,23\}$$

Finally, set

$$\gamma_{111],1} = \alpha'_{111],1} - \frac{1}{3}\beta'_{111],1},\ \ \gamma_{21],1} = \alpha'_{21],1} + \frac{2}{3}\beta'_{21],1},\ \ \gamma_{21],2} = \alpha'_{21],2} + \frac{1}{6}\beta'_{21],2},\ \ \gamma_{3],3} = \alpha'_{3],3},$$

$$\delta_{111],1} = \alpha'_{111],1} + \frac{1}{6}\beta'_{111],1},\ \ \delta_{21],1} = \alpha'_{21],1} - \frac{1}{3}\beta'_{21],1},\ \text{and}\ \delta_{21],2} = \alpha'_{21],2} - \frac{1}{3}\beta'_{21],2}$$

Thus,

$$\varphi_1(w) = \sum_{\substack{\lambda\in\Pi(3)\\ k\in\lambda^\circ}} \left[\gamma_{\lambda,k} \sum_{\substack{(S,Q)\in ECL\\ S\ni 1,|S|=k\\ Q\in Q_\lambda}} w(S,Q) + \delta_{\lambda,k} \sum_{\substack{(S,Q)\in ECL\\ S\not\ni 1,|S|=k\\ Q\in Q_\lambda}} w(S,Q)\right]$$

We should mention that a similar formula for linear and symmetric solutions of games in partition function form was obtained by Hernández-Lamoneda et al. [6].

Corollary 1. The space of all linear and symmetric solutions on $G^{(3)}$ has dimension

$$\left|\left\{\gamma_{\lambda,k} \mid \lambda \in \Pi(3), k \in \lambda^{\circ}\right\} \cup \left\{\delta_{\lambda,k} \mid \lambda \in \Pi(3) \setminus \{[3]\}, k \in \lambda^{\circ}\right\}\right| = 7$$

Once we have such a global description of all linear symmetric solutions, we can understand restrictions imposed by other conditions or axioms, for example, the efficiency axiom.

Proposition 5. Let $\varphi : G^{(3)} \to R^3$ be a linear symmetric solution. Then φ is efficient if and only if

1) $\varphi_i(u_\lambda^k) = 0$, for all $\lambda \in \Pi(3) \setminus \{[3]\}$ and all $k \in \lambda^{\circ}$; and

2) $\varphi_i(u_{[3]}^3) = \dfrac{1}{3}$

Proof. First of all, $\left(U_{[3]}^3\right)^{\perp}$ is exactly the subspace of games w where $w(N, \{N\}) = 0.$ Of these games, those in V_G trivially satisfy $\sum_{i \in N} \varphi_i(w) = 0,$ since (by Schur's Lemma) $\varphi(V_G) \subset V$.

Thus, efficiency need only be checked in U_G. Since u_λ^k is fixed by every permutation in S_3, we have

$$\sum_{i \in N} \varphi_i(u_\lambda^k) = 3\phi_i(u_\lambda^k)$$

and so φ is efficient if and only if $3\varphi_i(u_\lambda^k) = u_\lambda^k(N, \{N\}) = 1$ (if $\lambda = [3]$).

Recall that U_G is a subspace of games whose value for a given embedded coalition (S, Q) depends only on the cardinality of S and the structure of Q. The next corollary characterizes the solutions to these games in terms of linearity, symmetry and efficiency. It turns out that among all linear symmetric solutions, the egalitarian solution is characterized as the unique efficient solution on U_G. Formally:

Corollary 2. Let $\varphi : G^{(3)} \to R^3$ be a linear, symmetric and efficient solution. Then for all $w \in U_G$

$$\varphi_i(w) = \frac{w(N, \{N\})}{3}$$

In other words, all linear, symmetric and efficient solutions (e.g., Myerson's value) coincide with the egalitarian solution when restricted to these type of games, U_G.

Now, another immediate application is to provide a characterization of all linear, symmetric and efficient solutions[7].

Theorem 2. The solution $\varphi : G^{(3)} \to R^3$ satisfies linearity, symmetry and efficiency axioms if and only if it is of the form

$$\varphi_i(w) = \frac{w(N,\{N\})}{3} + \sum_{\substack{\lambda\in\Pi(3)\\ k\in\lambda^\circ}} \delta_{\lambda,k}\left[\sum_{\substack{(S,Q)\in ECL\\ S\ni i,|S|=k\\ Q\in Q_\lambda}}(n-k)w(S,Q) - \sum_{\substack{(S,Q)\in ECL\\ S\not\ni i,|S|=k\\ Q\in Q_\lambda}} kw(S,Q)\right] \tag{4}$$

for some real numbers $\left\{\delta_{\lambda,k} \mid \lambda\in\Pi(3)\setminus\{[3]\}, k\in\lambda^\circ\right\}$.

Proof. Let $\varphi : G^{(3)} \to R^3$ be a linear, symmetric and efficient solution, and $w\in G^{(3)}$. Then, by Proposition 3, Schur's Lemma and Proposition 5:

$$\begin{aligned}\varphi_i(w) &= \sum_{\substack{\lambda\in\Pi(3)\\ k\in\lambda^\circ}} a_{\lambda,k}\varphi_i\left(u_\lambda^k\right) + \sum_{\substack{\lambda\in\Pi(3)\setminus\{[3]\}\\ k\in\lambda^\circ}} \varphi_i\left(z_{\lambda,k}^{\lambda,k}\right)\\ &= a_{[3],3}\varphi_i\left(u_{[3]}^3\right) + \sum_{\substack{\lambda\in\Pi(3)\setminus\{[3]\}\\ k\in\lambda^\circ}} \beta\left(z_{\lambda,k}\right)_i\\ &= \frac{w(N,\{N\})}{3} + \sum_{\substack{\lambda\in\Pi(3)\setminus\{[3]\}\\ k\in\lambda^\circ}} \beta'_{\lambda,k}\psi_i\left(w_{\lambda,k}\right)\end{aligned}$$

where $\beta'_{\lambda,k} = \beta_{\lambda,k}(-1)^{|\lambda|}k(|\lambda|-1)$. The result follows from substituting the values $\psi_i(w_{\lambda,k})$ grouping terms, and setting $\delta_{111],1} = -\frac{1}{6}\beta'_{111],1}$, $\delta_{21],1} = \frac{1}{3}\beta'_{21],1}$, and $\delta_{21],2} = \frac{1}{6}\beta'_{21],2}$.

Corollary 3. The space of all linear, symmetric and efficient solutions on $G^{(3)}$ has dimension

$$\left|\left\{\delta_{\lambda,k} \mid \lambda\in\Pi(3)\setminus\{[3]\}, k\in\lambda^\circ\right\}\right| = 3$$

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[7] An equivalent expression to (4) can be found in the paper by Hernández-Lamoneda et al. [6].

It is also possible to give an expression for all linear, symmetric and efficient solutions of TU games in a characteristic function form. Let

$$J^{(n)} = \{v : 2^N \to R \mid v(\varnothing) = 0\}$$

be the set of all TU games in characteristic function form with n players.

Corollary 4. The solution $\varphi : J^{(3)} \to R^n$ satisfies the linearity, symmetry and efficiency axioms if and only if it is of the form

$$\varphi_i(v) = \frac{v(N)}{3} + \sum_{\substack{S \varnothing N \\ i \in S}} \rho_s \frac{v(S)}{s} - \sum_{\substack{S \subset N \\ i \notin S}} \rho_s \frac{v(S)}{n-s}$$

for some real numbers $\{\rho_1,\ \rho_2\}$.

Proof. Take $w \in G$ such that $w(S, Q) = v(S)$ for all $(S, Q) \in ECL$ in Eq. (4).

4 The case *n* = 4

One may notice that all the previous results follow from the decomposition of the space of games into a direct sum of irreducible subspaces. In this part, we provide such a decomposition for four player games.

Example 2. For the case $N = \{1, 2, 3, 4\}$, $\dim G^{(4)} = 37$ and following from (1), it decomposes as follows:

$$G^{(4)} = G^1_{[1,1,1,1]} \oplus G^1_{[2,1,1]} \oplus G^2_{[2,1,1]} \oplus G^1_{[3,1]} \oplus G^3_{[3,1]} \oplus G^2_{[2,2]} \oplus G^4_{[4]}$$

Once again, we first obtain a decomposition of each subspace of games G^k_λ into irreducible subspaces and hence derive the decomposition of $G^{(4)}$. For this purpose, let $I_{\lambda,k}$ be a set such that

$$I_{\lambda,k} = \begin{cases} \lambda^\circ \setminus \{k\} & \text{if} \quad m_k = 1 \\ \lambda^\circ & \text{if} \quad m_k > 1 \end{cases}$$

For each $\lambda \in \Pi(n) \setminus \{[n]\}$, $k \in \lambda^{\circ}$ and $x \in V_n$, define the set of games in G_λ^k, $\{x^{\lambda,k,j} \mid j \in I_{\lambda,k}\}$ as follows:

$$x^{\lambda,k,j}(S,Q) = \begin{cases} \sum\limits_{\substack{T \in Q \\ |T|=j, T \neq S}} \sum\limits_{i \in T} x_i & \text{if} \quad Q \in Q_\lambda, \quad |S| = k \\ 0 & \text{otherwise} \end{cases}$$

Proposition 6. For $\lambda \in \Pi(4) \setminus \{[4]\}$,

$$G_\lambda^k = U_\lambda^k \oplus V_\lambda^k \oplus W_\lambda^k$$

where $U_\lambda^k = \langle u_\lambda^k \rangle$; U_4, $V_\lambda^k = \underset{j \in I_{\lambda,k}}{\oplus} \{x^{\lambda,k,j} \mid x \in V_4\}$ and neither any $\{x^{\lambda,k,j} \mid x \in V_4\}$; V_4; nor W_λ^k contains any summands isomorphic to either U_4 or V_4. The decomposition is orthogonal (See proof in the Appendix).

Remark 4. Proposition 6 does not quite give us a decomposition of G_λ^k into irreducible summands. The subspace U_λ^k is irreducible and V_λ^k is a direct sum of irreducible subspaces, whereas W_λ^k may or may not be irreducible (depending on λ and k). However, as we shall see, the exact nature of this subspace plays no role in the study of linear symmetric solutions, since it lies in the kernel of any such solution.

As in the case of three player games, set $U_G = \underset{\substack{\lambda \in \Pi(4) \\ k \in \lambda^{\circ}}}{\oplus} U_\lambda^k$. Once again, U_G is a subspace of games, whose values $w(S,Q)$ depend only on the cardinality of S and on the structure of Q. Set $V_G = \underset{\substack{\lambda \in \Pi(4) \setminus \{[4]\} \\ k \in \lambda^{\circ}}}{\oplus} V_\lambda^k$ and $W_G = \underset{\substack{\lambda \in \Pi(4) \setminus \{[4]\} \\ k \in \lambda^{\circ}}}{\oplus} W_\lambda^k$, then:

$$G^{(4)} = U_G \oplus V_G \oplus W_G$$

Corollary 5. If $\varphi : G^{(4)} \to R^4$ is a linear symmetric solution, then $\varphi(w) = 0$ for each $w \in W_G$.

Proof. Let $\varphi : G^{(4)} = U_G \oplus V_G \oplus W_G \to R^4 = U_4 \oplus V_4$ be a linear symmetric solution. Assume $X \subset W_G$ is an irreducible summand in the decomposition of W_G (even when we do not know the decomposition of W_G as a sum of irreducible subspaces, it is known that such a decomposition exists). Let p_1 and p_2 denote the orthogonal projections of R^4 onto U_4 and V_4, respectively. Now, $\varphi : G^{(4)} \to R^4 = U_4 \oplus V_4$ may be written as $\varphi = (p_1 \circ \varphi, p_2 \circ \varphi)$. Denote by $\iota : X \to G^{(4)}$ the inclusion. Then the restriction of φ to X may be expressed as

$$\varphi_{|X} = \varphi \circ \iota = (p_1 \circ \varphi \circ \iota,\, p_2 \circ \varphi \circ \iota)$$

Now, $p_1 \circ \varphi \circ \iota : X \to U_4$ and $p_2 \circ \varphi \circ \iota : X \to V_4$ are linear symmetric maps. Since X is not isomorphic to either of these two spaces, Schur's Lemma (see the Appendix for its statement) says that $p_1 \circ \varphi \circ \iota$ and $p_2 \circ \varphi \circ \iota$ must be zero. Since this is true for every irreducible summand X of W_G, φ is zero on all of W_G.

Remark 5. According to Proposition 6 and the previous result, in order to study linear symmetric solutions, one only needs to look at those games inside $U_G \oplus V_G$.

As we have already pointed out, in the case of four player games, we can also obtain characterizations of the class of linear and symmetric solutions, as well as of the class of linear, symmetric and efficient solutions. Once again, the key is the decomposition of $G^{(4)}$ into irreducible subspaces (Proposition 6), together with Shur's Lemma.

5. Concluding remarks

We have noted that the point of view we take in this article depends heavily on the decomposition of the space of n-player games into a direct sum of "special" subspaces. In the cases where $n = 3, 4$, it was decomposed as a direct sum of three orthogonal subspaces: a subspace containing a type of symmetric games, another subspace which we call V_G, and a subspace W_G, which only plays the role of the common kernel of every linear symmetric solution. Although V_G does not have a natural definition in terms of well known game theoretic considerations, it has a simple characterization in terms of vectors whose entries add up to zero.

Characterizations of solutions follow from such a decomposition in an very economical way. It remains an open challenge to obtain the general decomposition of $G^{(n)}$ into a direct sum of irreducible subspaces, since mathematically, the general case seems to have a much more complicated structure.

Although it is true that the characterization of these results could be proved without any explicit mention of representation theory with regard to symmetric groups, we feel that by doing that we would be withholding valuable information from the reader. This algebraic tool, we believe, sheds new light on the structure of the space of games in partition function form and their solutions. Part of the purpose of the present paper is to share this viewpoint with the reader.

Appendix

A reference for basic representation theory is Fulton and Harris [4]. Nevertheless, we recall all the basic facts that we need.

The symmetric group S_n acts on G via linear transformations (i.e., G is a representation of S_n). That is to say, there is a group homomorphism $\rho : S_n \to GL(G)$, where $GL(G)$ is the group of invertible linear maps in G. This relation is given by:

$$(\theta w)(S, Q) := [\rho(\theta)(w)](S, Q) = w[\theta^{-1}(S, Q)]$$

for every $\theta \in S_n$, $w \in G$ and $(S, Q) \in ECL$.

Definition 6. Let H be an arbitrary group. A representation of H is a homomorphism $\rho : H \to GL(X)$, where X is a vector space and $GL(X) = \{T : X \to X \mid T$ linear and invertible$\}$.

In other words, a representation of H is a map assigning to each element $h \in H$ a linear map $\rho(h) : X \to X$ that respects multiplication:

$$\rho(h_1 h_2) = \rho(h_1)\rho(h_2)$$

for all $h_1, h_2 \in H$.

One usually abuses notation and talks about the representation X without explicitly mentioning the homomorphism ρ. Thus, when applying the linear transformation corresponding to $h \in H$ to the element $x \in X$, we write hx rather than $(\rho(h))(x)$.

The space of payoff vectors, $\mathbb{R}^n$ is also an S_n representation:

$$\theta(x_1, x_2, ..., x_n) := [\rho(\theta)](x_1, x_2, ..., x_n) = (x_{\theta(1)}, x_{\theta(2)}, ..., x_{\theta(n)})$$

Definition 7. Let X_1 and X_2 be two representations of the group H.

• A linear map $T : X_1 \to X_2$ is said to be H equivariant if $T(hx) = hT(x)$ for every $h \in H$ and $x \in X_1$.

- X_1 and X_2 are said to be isomorphic H-representations, $X_1; X_2$, if there exists a H-equivariant isomorphism between them.

Thus, two representations that are isomorphic are, as far as all problems dealing with linear algebra on a group of symmetries, the same. They are vector spaces of the same dimension, where actions are seen to correspond according to a linear isomorphism.

Definition 8. A representation X is irreducible if it does not contain a nontrivial invariant subspace. That is to say, if $Y \subset X$ is also a representation of H (meaning that $hy \in Y,\ \forall h \in H$), then Y is either $\{0\}$ or all of X.

Proposition 7. For any representation X of a finite group H, there is a decomposition

$$X = X_1^{\oplus a_1} \oplus X_2^{\oplus a_2} \oplus \cdots \oplus X_j^{\oplus a_j}$$

where the X_i are distinct irreducible representations. This decomposition is unique, as are the X_i that occur and their multiplicities a_i.

This property is called “complete reducibility” and the extent to which the decomposition of an arbitrary representation into a direct sum of irreducible ones is unique is one of the consequences of the following:

Theorem 3 (Schur’s Lemma). Let X_1, X_2 be irreducible representations of a group H. If $T : X_1 \to X_2$ is H equivariant, then $T = 0$ or T is an isomorphism.

Moreover, if X_1 and X_2 are complex vector spaces, then T is unique up to multiplication by a scalar $\lambda \in \mathbf{C}$.

The previous theorem is one of the reasons why it is worth carrying around the group action when there is one. Its simplicity hides the fact that it is a very powerful tool.

Following Fulton and Harris [4], the only three irreducible representations of S_3 are the trivial U_3, the standard V_3 and alternating representation[8] U'. Thus, for an arbitrary representation X of S_3 we can write

$$X = U_3^{\oplus a} \oplus U'^{\oplus b} \oplus V_3^{\oplus c} \tag{5}$$

and there is a way to determine the multiplicities a, b and c, in terms of $\tau = (123)$ and $\sigma = (12)$, which generate S_3, c, for example, is the number of independent eigenvectors

[8]Here, this action is given by $\theta x = \mathrm{sgn}(\theta)x$, for $\theta \in S_3$ and $x \in \mathbf{R}$.

of τ with eigenvalue ω[9] whereas $a + c$ is the multiplicity of 1 as an eigenvalue of σ, and $b + c$ is the multiplicity of -1 as an eigenvalue of σ.

Proposition 2. For $\lambda \in \Pi(3) \setminus \{[3]\}$

$$G_\lambda^k = U_\lambda^k \oplus V_\lambda^k$$

where $U_\lambda^k = \langle u_\lambda^k \rangle; U_3$ and $V_\lambda^k = \{z^{\lambda,k} \mid z \in V_3\}; V_3.$ The decomposition is orthogonal.

Proof. We start by showing that G_λ^k has exactly 1 copy of U_3 and 1 copy of V_3 if $\lambda \in \Pi(3) \setminus \{[3]\}$.

It is clear that $B = \{u_{(S,Q)} \mid (S, Q) \in ECL\}$ form a basis for $G^{(3)}$, where

$$u_{(S,Q)}(T, P) = \begin{cases} 1 & \text{if } (T, P) = (S, Q) \\ 0 & \text{otherwise} \end{cases} \tag{6}$$

For $G^{(3)}$, it is easy to verify that $[\tau]_B$ has the characteristic polynomial $p(x) = (x-1)^4\left[(x-\omega)(x-\omega^2)\right]^3$ and $[\sigma]_B$ has the characteristic polynomial $p(x) = (x-1)^7(x+1)^3$. From these and (5), we have $c = 3$, $a + c = 7$ and $b + c = 3$. Thus

$$G^{(3)} = U_3^{\oplus 4} \oplus V_3^{\oplus 3}$$

This implies directly that if $\lambda \in \Pi(3) \setminus \{[3]\}$, then every G_λ^k has exactly 1 copy of U_3 and 1 copy of V_3, since $G_{[3]}^3 = Ru_{[n]}^n;\ U_3$ and $\dim G_\lambda^k = 3$.

Now, define the map $T_\lambda^k : R^n \to G_\lambda^k$ by $T_\lambda^k(z) = z^{\lambda,k}$. This map is an isomorphism between U_λ^k and U_3 (similarly, between V_λ^k and V_3), since it is linear, S_3-equivariant and 1–1. From Proposition 9, we obtain the decomposition $R^3 = U_3 \oplus V_3$. Thus, inside G_λ^k, we have the images of these two subspaces: $U_\lambda^k = T_\lambda^k(U_3)$ and $V_\lambda^k = T_\lambda^k(V_3)$.

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[9]Denoting by $1, \omega, \omega^2$ the cube roots of unity.

Finally, the invariant inner product, defines an equivariant isomorphism, which in particular must preserve the decomposition. This implies the orthogonality of the decomposition.

There is a remarkably effective technique for decomposing any given finite dimensional representation into its irreducible components. The secret is *character theory*. In the analysis of the representations of S_3, the key was to study the eigenvalues of the actions of individual elements of S_3. This is the starting point of character theory. Finding individual eigenvalues, however, is difficult. Luckily, it is sufficient to consider their sum, the trace, which is much easier to compute.

Definition 9. Let $\rho : H \to GL(X)$ be a representation. The character of X is the complex-valued function $\chi_X : H \to \mathbf{C}$, defined as:

$$\chi_X(h) = Tr\left(\rho(h)\right)$$

The character of a representation is easy to compute. If H acts on an n-dimensional space X, we write each element h as an $n \times n$ matrix according to its action expressed in some convenient basis, then sum up the diagonal elements of the matrix describing h to get $\chi_X(h)$. For example, the trace of the identity map of an n-dimensional vector space is the trace of the $n \times n$ identity matrix, i.e. n. In fact, $\chi_X(e) = \dim X$ for any finite dimensional representation X of any group.

Notice that, in particular, we have $\chi_X(h) = \chi_X(ghg^{-1})$ for $g, h \in H$. So that χ_X is constant on the conjugacy classes of H. Such a function is called a class function.

Definition 10. Let $\mathbf{C}_{\text{class}}(H) = \{f : H \to \mathbf{C} \mid f$ is a class function on $H\}$. If $\chi_1, \chi_2 \in \mathbf{C}_{\text{class}}(H)$, we define an Hermitian inner product on $\mathbf{C}_{\text{class}}(H)$ by

$$\langle \chi_1, \chi_2 \rangle = \frac{1}{|H|} \sum_{h \in H} \overline{\chi_1(h)}\, \chi_2(h) \tag{7}$$

As was said, the character of a representation of a group H is really a function on the set of conjugacy classes in H. This suggests expressing the basic information about the irreducible representations of a group H in the form of a character table. This is a table with the conjugacy classes $[h]$ of H listed across the top, usually given by a representative h, with the number of elements in each conjugacy class written above it. The irreducible representations of H are listed on the left and the value of the character on the conjugacy class $[h]$ is given in the appropriate cell. For example, if $H = S_4$ and we only focus on the irreducible representations U_4 and V_4, then[10]:

	1	6	8	6	3
S_4	$[e]$	$[(12)]$	$[(123)]$	$[(1234)]$	$[(12)(34)]$
U_4	1	1	1	1	1
V_4	3	1	0	−1	−1

Finally, the multiplicities of the irreducible subspaces in a representation can be calculated via the following proposition:

Proposition 9. If $Z = Z_1^{\oplus a_1} \oplus Z_2^{\oplus a_2} \oplus \cdots \oplus Z_j^{\oplus a_j}$, then the multiplicity of Z_i (irreducible representation) in Z, is:

$$a_i = \left\langle \chi_Z, \chi_{Z_i} \right\rangle$$

where, is the inner product given by (7).

Proposition 6. For $\lambda \in \Pi(4) \setminus \{[4]\}$,

$$G_\lambda^k = U_\lambda^k \oplus V_\lambda^k \oplus W_\lambda^k$$

where $U_\lambda^k = \left\langle u_\lambda^k \right\rangle$; U_4, $V_\lambda^k = \underset{j \in I_{\lambda,k}}{\oplus} \left\{ x^{\lambda,k,j} \mid x \in V_4 \right\}$ and neither any $\left\{ x^{\lambda,k,j} \mid x \in V_4 \right\}$; V_4; nor W_λ^k contains any summands isomorphic to either U_4 or V_4. The decomposition is orthogonal.

Proof. First, $\left\langle \chi_{G_\lambda^k}, \chi_{U_4} \right\rangle$ and $\left\langle \chi_{G_\lambda^k}, \chi_{V_4} \right\rangle$ are the number of subspaces isomorphic to the trivial (U_4) and standard representation (V_4) within G_λ^k, respectively. The characters for each G_λ^k are given by[11]:

[10]In fact, there are five irreducible representations of S_4.

[11]In which a convenient basis is the one given in (6).

	1	6	8	6	3
S_4	[(1)]	[(12)]	[(123)]	[(1234)]	[(12)(34)]
$G^1_{[1111]}$	4	2	1	0	0
$G^1_{[211]}$	12	2	0	0	0
$G^2_{[211]}$	6	2	0	0	2
$G^1_{[31]}$	4	2	1	0	0
$G^3_{[31]}$	4	2	1	0	0
$G^2_{[22]}$	6	2	0	0	2

Thus from (7), $\left\langle \chi_{G^k_\lambda}, \chi_{U_4} \right\rangle = 1$ for each G^k_λ,

$$\left\langle \chi_{G^1_{[1111]}}, \chi_{V_4} \right\rangle = \left\langle \chi_{G^2_{[211]}}, \chi_{V_4} \right\rangle = \left\langle \chi_{G^1_{[31]}}, \chi_{V_4} \right\rangle = \left\langle \chi_{G^3_{[31]}}, \chi_{V_4} \right\rangle = \left\langle \chi_{G^2_{[22]}}, \chi_{V_4} \right\rangle = 1$$

and

$$\left\langle \chi_{G^1_{[211]}}, \chi_{V_4} \right\rangle = 2$$

The last part is to identify such copies of U_4 and V_4 inside G^k_λ. For this end, define the functions $L_{\lambda,k,j} : R^n \to G^k_\lambda$ by $L_{\lambda,k,j}(x) = x^{\lambda,k,j}$. These maps are isomorphisms between U^k_λ and U_4 (similarly, between $\left\{ x^{\lambda,k,j} \mid x \in V_4 \right\}$ andV_4), since they are linear, S_4-equivariant and 1–1. Thus, inside of G^k_λ, we have the following images of these two subspaces: $U^k_\lambda = L_{\lambda,k,j}(U_4)$ and $V^k_\lambda = \bigoplus_{j \in I_{\lambda,k}} L_{\lambda,k,j}(V_4)$.

The orthogonality of the decomposition follows again from the fact that the invariant inner product $\langle , \rangle$ gives an equivariant isomorphism, which preserves the decomposition.

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