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Elegant Hermite–Gaussian and Laguerre–Gaussian beams at a dielectric interface

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Basic features of narrow optical beam interactions with a dielectric interface are analysed. It is shown that two types of paraxial beams – elegant Hermite–Gaussians of linear polarization and elegant Laguerre–Gaussians of circular polarization – can be treated as vector normal modes of the interface. Excitation of higher-order modes by cross-polarization coupling is described and changes of mode indices induced by transmission and reflection matrices are evaluated. Optical vortex excitation and splitting are theoretically predicted and confirmed by numerical simulations.

Keywords: Hermite–Gaussian and Laguerre–Gaussian beams, dielectric interfaces, total internal reflection, optical vortices and cross-polarization coupling.

1. Introduction

Two basic families of solutions of the paraxial wave equation, in a form of Hermite–Gaussian (HG) beams and Laguerre–Gaussian (LG) beams, constitute two complete infinite-dimensional bases for a beam field. There are also two basic types, linear and circular, of beam polarization. For beams of uniform polarization, propagating in a homogeneous isotropic medium, beam polarization is decoupled from beam field distribution. However, for beams interacting with a planar discontinuity of a medium, the beam polarization and spatial characteristics are interrelated by cross-polarization coupling (XPC) at this interface [1, 2]. This coupling favours a special type of the HG and LG beams – the so-called elegant EHG and ELG beams, as described some time ago by SIEGMAN [3]. These beams, or strictly the EHG beams with linear TE or TM polarization and the ELG beams with circular left-handed (CL) and right-handed (CR) polarization, both of them of arbitrary order, can be considered as vector normal modes at the interface [4]. Main features of such narrow beams, of the order of one wavelength in their radius, reflected at a homogeneous, isotropic and lossless dielectric interface, are theoretically analysed and presented by a few examples of numerical simulations.

The field distribution of the reflected beam component of the same polarization as that of the incident beam basically mimics the field distribution of the incident beam. However, in the opposite (orthogonal) beam component, the XPC effect leads to the excitation of higher-order or lower-order beam modes. This process critically depends on beam polarization, phase and intensity distribution as well as on a propagation direction of the incident beam [4, 5]. Theoretical details of analysis of the problem were more extensively explained in [1, 4]. Note only here that the beam reflection process is determined by new reflection/transmission coefficients expressed by sums and differences of the *p* and *s* Fresnel coefficients.

2. Normal modes at the interface

For any incident normal mode, a total optical field at the interface resolves only into three beams – the incident, reflected and transmitted beams, with scalar normal modes attributed to each polarization component of these beams. Two sets of such scalar normal modes – the EHG beams and the ELG beams – will be considered in this analysis within the paraxial approximation of the wave equation. On their grounds the vector modes of the interface can be defined. They specify solutions to the problem in two coordinate systems: the Cartesian coordinate system $OXYZ$ of rectangular symmetry with coordinates X and Y placed in the interface plane; and the complex coordinate system $O\varsigma\bar{\varsigma}Z$ with a complex coordinate $\varsigma = 2^{-1/2}(X + iY)$ and its complex conjugate $\bar{\varsigma}$. It appears that the elegant beams with their possible non-paraxial extensions are particularly suitable in description of beam fields at the interface, provided that they are defined at the interface plane $X-Y$ instead, as usual, in the beam transverse plane $x-y$ (see Fig. 1).

Let us consider first the case of normal incidence and define the fundamental Gaussian beam field distribution by:

$$G(X, Y, Z) = \left(\frac{w_w}{v(Z)}\right)^2 \exp\left[\frac{-\frac{1}{2}(X^2 + Y^2)}{v^2(Z)}\right] \tag{1}$$

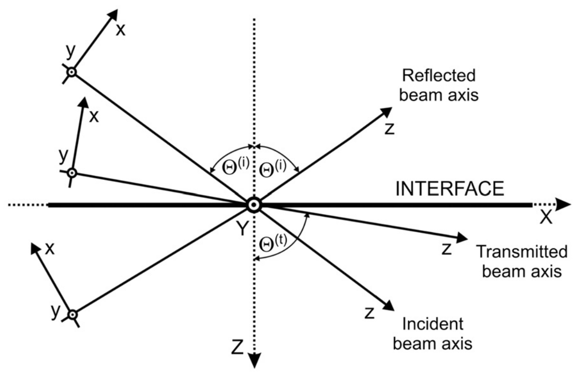

Fig. 1. Interface $OXYZ$ and beam $Oxyz$ reference frames viewed in a main incidence plane $X-Z$ in the case of internal reflection and with $\Theta^{(i)}$ and $\Theta^{(t)}$ denoting the incident and refracted angles, respectively.

where $z_D = kw_w^2$, k and w_w are the diffraction length, the wave number and the waist radius, respectively, and $v(Z) = w_w(1 + iZ/z_D)^{1/2}$ is the complex radius of the fundamental Gaussian mode $G(X, Y, Z)$. This mode is assumed here as symmetric with a circular cross-section at the interface, *i.e.*, with the same waist radii $w_{wX} = w_{wY} = w_w$ in X and Y directions in the plane $Z = 0$. For an elliptic cross-section of the beam further modifications into the case $w_{wX} \neq w_{wY}$ are straightforward. The elegant beams of arbitrary order can be then directly defined at the interface by appropriate differentiation of the fundamental Gaussian (1) at the interface.

For the EHG beams, the differentiation is performed with respect to the coordinates X and Y and the beams are specified by X and Y and indices m and n, respectively:

$$G_{m,n}^{(\mathrm{EH})}(X, Y, Z) = (w_w)^{m+n} \partial_X^m \partial_Y^n G(X, Y, Z) \tag{2}$$

Similarly, for ELG beams, the differentiation is performed with respect to the coordinates ς and $\bar{\varsigma}$ and the beams are specified by the radial and azimuthal indices p and l, respectively:

$$G_{p,l}^{(\mathrm{EL})}(\varsigma, \bar{\varsigma}, Z) = (w_w)^{2p+l} \partial_\varsigma^p \partial_{\bar{\varsigma}}^{p+l} G(X, Y, Z) \tag{3}$$

As a result, the EHG beams of the order $m + n$ are expressed by the Hermite polynomials $H_m(x)$ and $H_n(y)$:

$$G_{m,n}^{(\mathrm{EH})}(X, Y, Z) = (-w_w)^{m+n} H_m\left(\frac{2^{-1/2}X}{v(Z)}\right) H_n\left(\frac{2^{-1/2}Y}{v(Z)}\right) G(X, Y, Z) \tag{4}$$

and, similarly, the ELG beams of the order $2p + l$ are expressed by the associated Laguerre polynomials $L_p^l(x)$ of the order:

$$G_{p,l}^{(\mathrm{EL})}(\varsigma, \bar{\varsigma}, Z) = (-1)^{p+l}\left(\frac{\varsigma_\perp}{v}\right)^l \left(\frac{v}{w_w}\right)^{-(2p+l)} p!L_p^l\left(\frac{\varsigma_\perp^2}{v^2}\right) G_{0,0}^{(\mathrm{EL})}(\varsigma, \bar{\varsigma}, Z)\exp(il\psi) \tag{5}$$

where $\varsigma = \varsigma_\perp \exp(i\psi)$ with $\varsigma_\perp$ and ψ real.

The definitions (3) and (4), given here for beam normal incidence, follow exactly the well known definitions of elegant beams propagating in free space, as introduced by Siegman [3]. On the other hand, these definitions remain also valid in the case of oblique incidence after introducing the replacement of the "interface" coordinates X, Y and Z by the "beam" coordinates x, y and z (*cf.* Fig. 1). However, such replacement does not lead to the definition of the beam modes at the interface, as for them the condition (for one incident mode only one reflected mode and one transmitted mode of the same type (here EHG or ELG) are excited in each mode polarization component) is not exactly fulfilled.

In order to satisfy this condition, it is further stipulated that the beam mode definitions (2) and (3) still remain valid for oblique incidence, provided however that only the Gaussian beam $G(X, Y, Z)$ in Eqs. (2) and (3) takes the common form given in the coordinates x, y and z instead of X, Y and Z and, in addition, that all these mode fields acquire the additional phase shift $\Phi(X) = k^{(i)} X \sin\theta^{(i)}$ at the interface plane $Z = 0$. This is the result of the tilted phase fronts of the beams with respect to the interface plane:

$$G_{m,n}^{(\mathrm{EH})}(X, Y, Z) \to G_{m,n}^{(\mathrm{EH})}(X, Y, Z)\exp\left(ik^{(i)} X \sin\theta^{(i)}\right) \tag{6}$$

$$G_{p,l}^{(\mathrm{EL})}(\varsigma, \bar{\varsigma}, Z) \to G_{p,l}^{(\mathrm{EL})}(\varsigma, \bar{\varsigma}, Z)\exp\left(ik^{(i)} 2^{-1/2}(\varsigma + \bar{\varsigma})\sin\theta^{(i)}\right) \tag{7}$$

It appears from the analytic form of the transmission and reflection matrices (given with detailed explanation in [1, 4]) that, for oblique incidence, the modes such defined are in fact normal modes at the interface as they satisfy the above condition: for one incident mode only one reflected mode and one transmitted mode of the same type.

2.1. Reflection of the elegant Hermite–Gaussian modes

The action of the interface on any incident beam field can be more directly described in its spectral domain [1, 4]. It appears that, for any parallel (to the interface plane) spectral component $\underline{\tilde{E}}^{(i)} = \left[\tilde{E}_X^{(i)}, \tilde{E}_Y^{(i)}\right]$ of the incident EHG beam mode $\tilde{G}_{X,Y}^{(\mathrm{EH})}$, given at the interface plane X–Y with spectral amplitudes $\tilde{a}_X$ and $\tilde{a}_Y$ and of an arbitrary polarization parameter $\tilde{\chi}_{(X,Y)}^{(i)} = \tilde{a}_X / \tilde{a}_Y$:

$$\underline{\tilde{E}}^{(i)} = \begin{bmatrix} \tilde{a}_X \\ \tilde{a}_Y \end{bmatrix} \tilde{G}_{m,n}^{(\mathrm{EH})} \tag{8}$$

a spectral component of the reflected beam field

$$\begin{bmatrix} -\tilde{E}_X^{(r)} \\ \tilde{E}_Y^{(r)} \end{bmatrix} \cong \begin{bmatrix} r_p \tilde{a}_X \\ r_s \tilde{a}_Y \end{bmatrix} \tilde{G}_{m,n}^{(\mathrm{EH})} - 2r_{CX}(k_\perp w_w)^{-2} \begin{bmatrix} \tilde{a}_Y \\ -\tilde{a}_X \end{bmatrix} \tilde{G}_{m',n+1}^{(\mathrm{EH})} \tag{9}$$

is determined by the Fresnel coefficients r_p and r_s, together with the new XPC coefficient $r_{CX} = 0.5(r_p + r_s)$ describing the cross-polarization coupling between the TM and TE beam components. The choice of one value of m' from two possibilities $m' = 1$ or $m' = 0$ will be explained below. The action of the interface under normal incidence of the narrow ($kw_w = 2\pi$) EHG beam is graphically described by numerical simulations shown in Fig. 2.

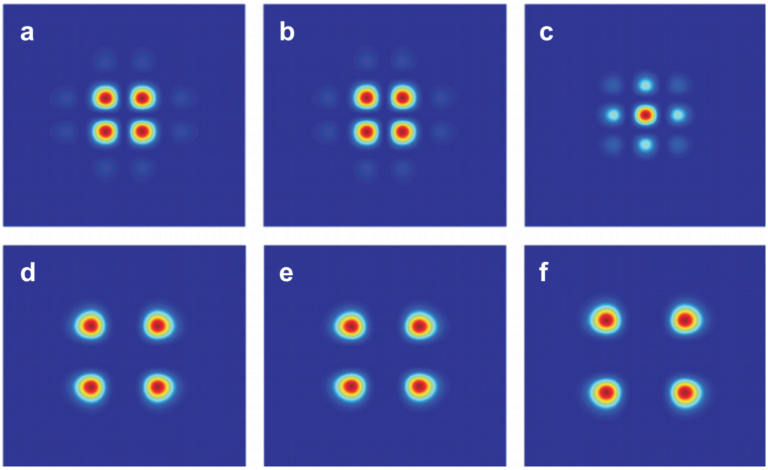

Fig. 2. Intensity transverse distribution of the EHG beam components in the configuration (**a**–**c**) and spectral (**d**–**f**) domains; the incident $EHG_{3,3}$ beam of TE polarization (**a**, **d**), the reflected $EHG_{3,3}$ TE beam component (**b**, **e**), the reflected $EHG_{4,4}$ TM beam component (**c**, **f**).

For normal incidence of the $EHG_{m,n}$ beam mode of TE or TM polarization, the reflected beam mode of the same polarization preserves the values of the incident mode indices m and n (*cf.* Figs. 2**b** and 2**e**). However, the reflected beam mode of the opposite (TM or TE) polarization acquires the change by +1 in both their mode indices: $m \to m' = m + 1$ and $n \to n + 1$ (*cf.* Figs. 2**c** and 2**f**). Note also that for beam oblique incidence only the change $m \to m' = m$ and $n \to n + 1$ takes place, as one from these two index changes is gradually replaced by a net longitudinal shift of a whole beam structure in the incidence plane X–Z [4].

2.2. Reflection of the elegant Laguerre–Gaussian modes

Similarly, for any parallel (to the interface plane) spectral component $\underline{\tilde{E}}^{(i)} = \left[\tilde{E}_R^{(i)}, \tilde{E}_L^{(i)}\right]$ of the incident ELG beam mode given at the interface plane X–Y with spectral amplitudes $\tilde{a}_R$ and $\tilde{a}_L$ and of an arbitrary polarization parameter $\tilde{\chi}_{(R,L)}^{(i)} = \tilde{a}_R / \tilde{a}_L$:

$$\underline{\tilde{E}}^{(i)} = \begin{bmatrix} \tilde{a}_R \\ \tilde{a}_L \end{bmatrix} \tilde{G}_{p,l}^{(\mathrm{EL})} \tag{10}$$

a spectral component of the reflected beam field:

$$\begin{bmatrix} \tilde{E}_L^{(r)} \\ \tilde{E}_R^{(r)} \end{bmatrix} = r_C \begin{bmatrix} \tilde{a}_R \\ \tilde{a}_L \end{bmatrix} \tilde{G}_{p,l}^{(\mathrm{EL})} + r_{CX} \begin{bmatrix} \tilde{a}_L \, \tilde{G}_{p+1,l-2}^{(\mathrm{EL})} \\ \tilde{a}_R \tilde{G}_{p-1,l+2}^{(\mathrm{EL})} \end{bmatrix} \tag{11}$$

is determined through the Fresnel coefficients r_p and r_s by the XPC reflection coefficient $r_{CX} = 0.5(r_p + r_s)$, together with the new "direct" coefficient CR. Note that in (11) the sign (CR and CL) of beam circular polarization is defined with respect to $-Z$ direction and that the field vector $\underline{\tilde{E}}^{(r)} = \left[\tilde{E}_L^{(r)}, \tilde{E}_R^{(r)}\right]$ should be replaced by $\underline{\tilde{E}}^{(r)} = \left[\tilde{E}_R^{(r)}, \tilde{E}_L^{(r)}\right]$ for incidence angles greater than the Brewster angle. The action of the interface on the ELG beam under normal incidence is graphically described by numerical simulations shown in Fig. 3.

For normal incidence of the $\mathrm{ELG}_{p,\,l}$ beam mode of CR or CL polarization, the reflected beam mode of the CL or CR polarization preserves the incident mode

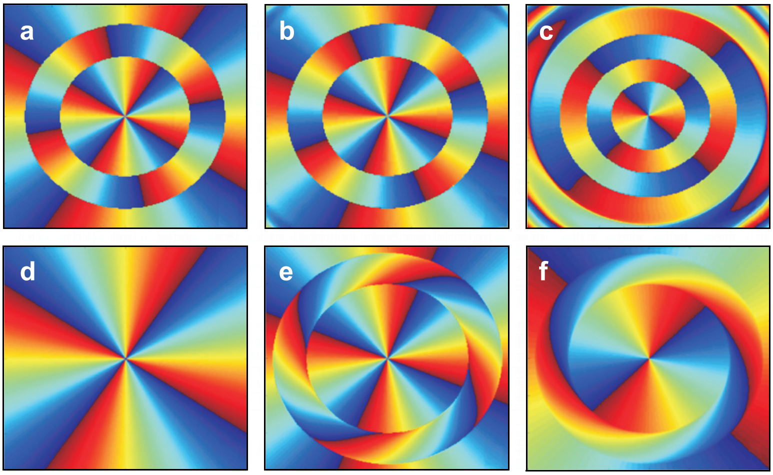

Fig. 3. Phase transverse distribution of the ELG beam components in the configuration (**a**–**c**) and spectral (**d**–**f**) domains, the case of normal incidence; the incident $\mathrm{ELG}_{2,\,4}$ beam of CL polarization (**a**, **d**), the reflected $\mathrm{ELG}_{2,\,4}$ CR beam component (**b**, **e**), the reflected $\mathrm{ELG}_{3,\,2}$ CL beam component (**c**, **f**).

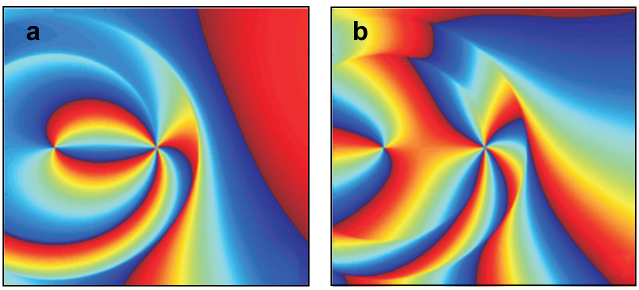

Fig. 4. Optical vortex excitation and splitting induced by the XPC effect for the incident beam of the $\mathrm{ELG}_{2,\,4}$ shape. Phase distribution of the reflected beam mode is shown in the spectral domain: for the CL polarization of the incident beam mode – the total topological charge of the reflected beam CR component equals 4 – 2 (**a**), for the CR polarized incident beam mode – the total topological charge of the reflected beam CL component equals 4 + 2 (**b**).

indices p and l (*cf.* Figs. 3**b** and 3**e**). However, the reflected beam mode of the opposite, CR or CL polarization, acquires the change by ±2 in their azimuthal indices, meanwhile their radial indices are changed by ∓ 1 ; $p \to p \mp 1$ and $l \to l \pm 2$ (Figs. 3**c** and 3**f**). Therefore, although a mode order $2p + 1$ remains unchanged, excitation of optical vortices is observed. These changes are opposite in sign for the opposite polarization (CL or CR) components of the incident beam (*cf.* Fig. 4).

Similar changes augmented by vortex splitting can be also observed for oblique incidence of the sufficiently narrow ELG beams of CL or CR polarization; however, in addition to the excited vortex splitting in the reflected beam component excited by the XPC effect [4]. Its polarization (CR or CL) depends on whether the incidence angle is smaller or greater than the Brewster angle; see also the comment given below Eq. (11). An example of the phenomenon of vortex excitation and splitting and their explicit dependence on beam polarization is shown in Fig. 4 after [6], where the reflection and transmission phenomena of the EHG and ELG beams at the interface have been more closely described and discussed.

3. Conclusions

Two sets of elegant higher-order beams have been considered as vector normal modes of the interface: the EHG beams of linear polarization and the ELG beams with circular polarization, with some modifications necessary in the case of oblique incidence. Excitation of the opposite (orthogonal) components of the reflected mode field, as well as changes in their mode spatial structures under reflection at the interface (similarly under refraction) are caused by the XPC effect and depend on polarization of the incident beam and its incidence angle.

For normal incidence, the EHG modes, the TM or TE components of the reflected field excited by the XPC effect acquire changes by +1 in both their mode indices. In the case of the incident ELG modes with CR or CL polarization, their radial indices are changing by ∓ 1 and their azimuthal indices are changing by ±2 for the CR or CL components of the reflected field excited by the XPC effect. The last changes indicate vortex excitation at the interface resulting in different topological charges of different polarization components of the reflected ELG modes.

For oblique incidence and with increasing the incidence angle, the longitudinal index changes of the EHG beams are gradually replaced by the net longitudinal spatial displacement of the whole field structure. For sufficiently narrow ELG beams vortex splitting is observed in addition to their excitation under oblique incidence. A topological charge of the excited vortex adds to or subtracts from a charge of the vortex driven by the incident beam.

Note that the idea of the XPC beam-interface interactions was explicitly introduced in [7] and subsequently applied in evaluation of field distribution of the first-order [8] or all higher-order [1, 2] modes excited by the XPC effects at the interface. Further analysis of this problem can be found, for example, in [4, 5, 9, 10].

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